Spectral pseudorandomness and the clique number of the Paley graph

Tim Kunisky

Yale University

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I. Introduction

Paley Graph

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$\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ = finite field on p elements for $p \equiv 1 \bmod 4$.

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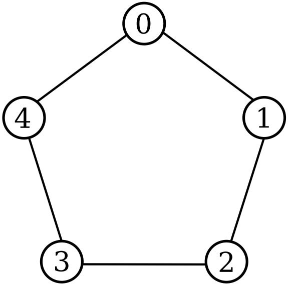
Example: $p = 5 \rightsquigarrow$ squares are $\{1, 4 \equiv -1\}$.



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$\Longrightarrow$ adjacencies in G_p look independent.

$\Longrightarrow$ G_p is **pseudorandom**, behaving like Erdős-Rényi graph with edge probability $\frac{1}{2}$ (since $\deg(x) = \frac{p-1}{2} \sim \frac{1}{2}p$).

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Example: As $p \to \infty$,

$$\begin{aligned}\#\text{ triangles in } G_p &\sim \mathbb{E}\left[\#\text{ triangles in ER}\right] \\ &= \binom{p}{3}\left(\frac{1}{2}\right)^3 \sim \frac{1}{48}p^3.\end{aligned}$$

Paley Graphs: The Clique Number

Question: How about extremal questions (large subgraphs)?

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Same for $\omega(G_p)$? Not quite...

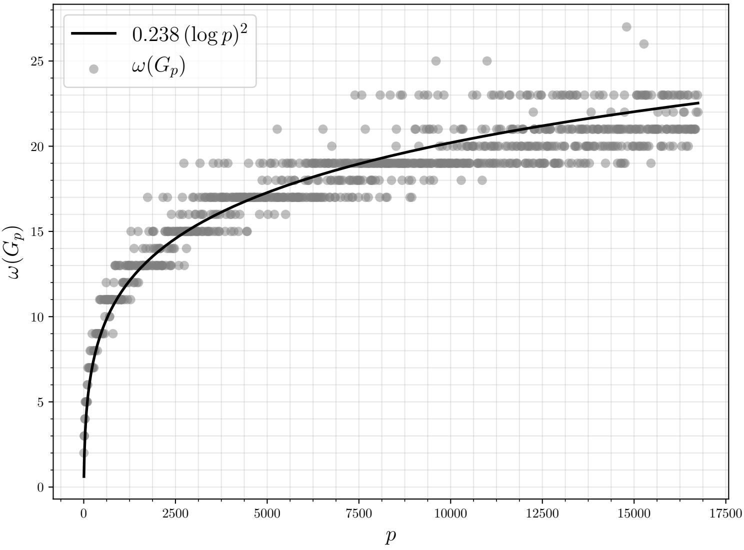
$$\omega(G_{p_i}) \geq \log p_i \log\log\log p_i \qquad \text{[Graham, Ringrose '90]}$$

$$\omega(G_p) \overset{?}{\sim} (\log p)^2 \qquad \text{(numerics)}$$

And, in any case, the best **upper bounds** we have are

$$\omega(G_p) \leq \sqrt{p} \qquad \text{(spectral/Hoffman/trivial bound)}$$

$$\omega(G_p) \leq \sqrt{p/2} + 1 \qquad \text{[Hanson, Petridis '21]}$$



$0.238\,(\log p)^2$
$\omega(G_p)$
$\omega(G_p)$
p
0
5
10
15
20
25
0
2500
5000
7500
10000
12500
15000
17500

Big **number theory** question:

What proof technique can break the "square root barrier" and prove

$$\omega(G_p) = O(p^{1/2-\varepsilon}) \ ?$$

II. Sum-of-Squares Relaxations

(joint work with with Xifan Yu)

A degree 4 sum-of-squares lower bound for the clique number of the Paley graph [arXiv:2211.02713]

Sum-of-Squares (SOS) Relaxations

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$$\omega(G) = \max \left\{ \sum_{i \in V} y_i \;:\; y_i^2 - y_i = 0, \quad y_i y_j = 0 \text{ if } \{i, j\} \notin E \right\}$$

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3. Optimize $\sum_{i \in V} X(\{i\})$ over that enlarged set

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$d \geq 2 \rightsquigarrow \mathsf{SOS}_{2d}(G) \geq \omega(G)$, tighter bounds in time $p^{O(d)}$.

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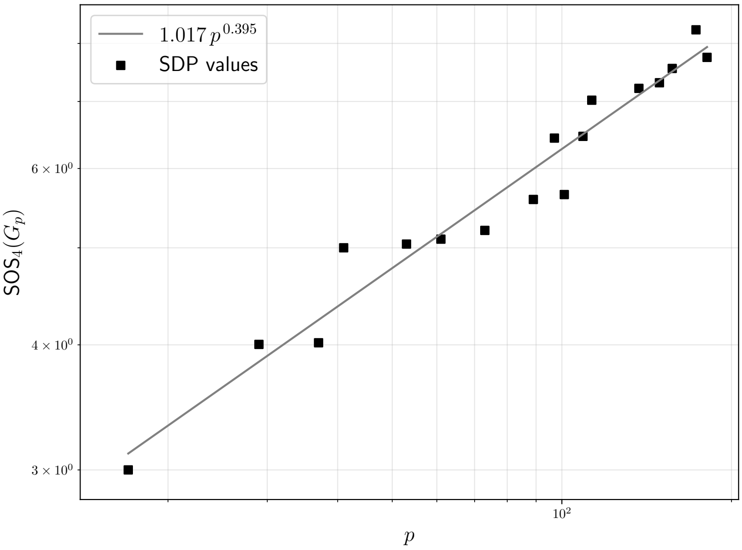
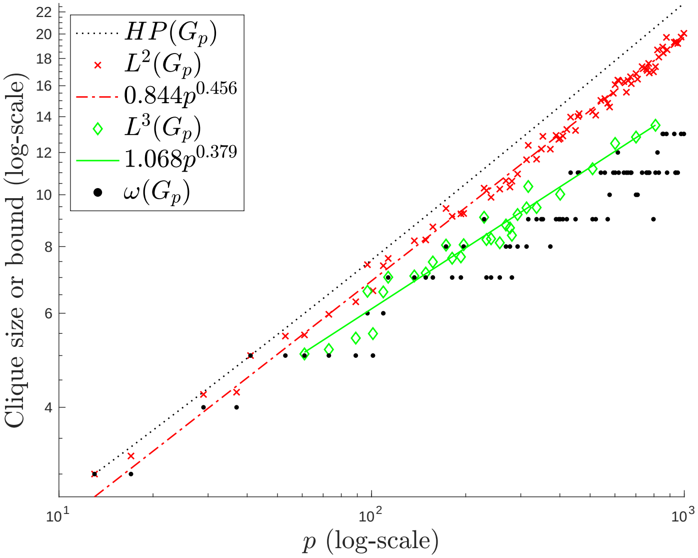
Question: Does this transfer to Paley graphs, showing that low-degree SOS cannot break the $\sqrt{p}$ barrier?



$1.017\,p^{0.395}$
SDP values
6×10^0
4×10^0
3×10^0
$\mathsf{SOS}_4(G_p)$
10^2
p



[Gvozdenović, Laurent, Vallentin ’09; Kobzar, Mody ’23 (forthcoming)]

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1. Derandomizes an early result on the random graph case: [DM '15] showed $\mathbb{E}[\mathsf{SOS}_4(\mathsf{ER})] = \widetilde{\Omega}(p^{1/3})$.
2. Compatible with numerics: maybe $\mathsf{SOS}_4(G_p) \sim p^{0.4}$.

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Theorem: [KY '22] For Paley graphs, such proves only

$$\mathsf{SOS}_4(G_p) = \Omega(p^{1/3}),$$

i.e., our main result cannot be improved without a fancier choice of $\boldsymbol{X}$ $\rightsquigarrow$ probably significantly harder to analyze.

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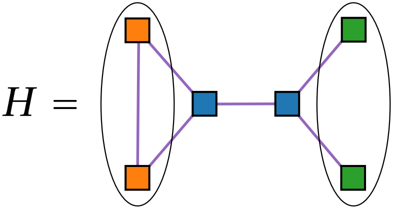
Example: For a graph with sets of "left" and "right" vertices



we get a matrix

$$M^H(G)_{(a,b),(c,d)} = \sum_{i \neq j \notin \{a,b,c,d\}} A_{a,b} A_{a,i} A_{b,i} A_{i,j} A_{j,c} A_{j,d}.$$

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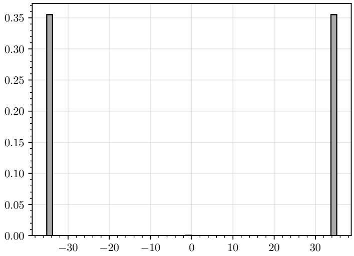
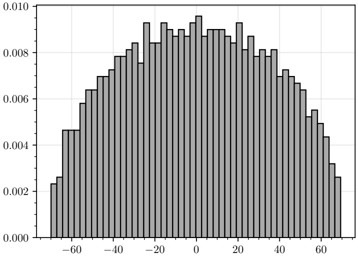
Basically, can build these by taking advantage of the discrepancy between

$$\boldsymbol{A}_{G_p}^2 = p\boldsymbol{I} - \boldsymbol{1}\boldsymbol{1}^\top,$$
$$\boldsymbol{A}_{\mathsf{ER}}^2 = p\boldsymbol{I} + \sqrt{p} \cdot (\text{random matrix}).$$

Our intuition: If SOS breaks the square root barrier, it is thanks to a spectral failure of pseudorandomness:

$$\lambda(G_p) \qquad \not\approx \qquad \lambda(\mathsf{ER})$$





Proof Idea

Also boils down to bounding $\|\boldsymbol{M}^H(G_p)\|$ for various H using $\operatorname{Tr} \boldsymbol{M}^H(G)^k$, but with different tools.

[AMP '16], [BHKKMP '19]: **combinatorics** from $\mathbb{E}[\operatorname{Tr} \boldsymbol{M}^H(\mathsf{ER})^k]$

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For $\chi : \mathbb{F}_p \to \mathbb{C}$ the **Legendre symbol** character,

$$(\boldsymbol{A}_{G_p})_{i,j} = \left\{ \begin{array}{ll} +1 & \text{if } i \sim j \\ -1 & \text{if } i \not\sim j \end{array} \right\} = \chi(i - j),$$

so polynomials in χ appear in entries of $\boldsymbol{M}^H$. Not many good tools for handling $\operatorname{Tr} \boldsymbol{M}^H(G_p)^k$ character sums, but we can use other case-by-case tricks to mostly avoid these.

Character Sum Estimates

Typical, more classical, univariate example:

Theorem: (Weil) If $f \in \mathbb{F}_p[x]$ is not a multiple of a perfect square, then

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But we need the **much harder** multivariate case:

$$\left| \sum_{a_1, \ldots, a_k \in \mathbb{F}_p} \chi(f(a_1, \ldots, a_k)) \right| \overset{?}{\lesssim} \sqrt{p^k}.$$

III. Spectral Pseudorandomness

Generic MANOVA limit theorems for products of projections
[arXiv:2301.09543]

Next: How (spectrally) pseudorandom is G_p, if at all? Can we use this to prove clique number bounds?

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Defining $G_{p,\{0\}}$:= induced subgraph on $\{i : i \sim 0 \text{ in } G_p\}$,

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Why stop there? G_p is also **edge transitive**, so

$$\omega(G_p) = 2 + \omega(G_{p,\{0,1\}}).$$

The Localization Approach: Formulas [MMP '19]

G_p is **vertex transitive**, so there is a maximum clique that contains $0 \in \mathbb{F}_p$.

Defining $G_{p,\{0\}}$:= induced subgraph on $\{i : i \sim 0 \text{ in } G_p\}$,

$$\omega(G_p) = 1 + \omega(G_{p,\{0\}}).$$

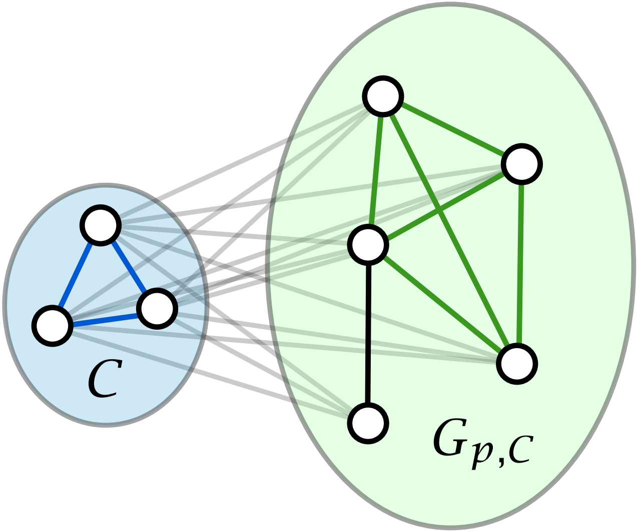
Why stop there? G_p is also **edge transitive**, so

$$\omega(G_p) = 2 + \omega(G_{p,\{0,1\}}).$$

Why stop there? We don't need transitivity; for any k,

$$\omega(G_p) = k + \max_{C \text{ a } k\text{-clique in } G_p} \omega(G_{p,C}).$$

Local Graphs



The Localization Approach: Bounds [MMP '19]

Now, can plug in our favorite clique number bounds and try to control those. [MMP '19] found empirically

$$\omega(G_p) \leq 1 + \mathsf{SOS}_2(G_{p,\{0\}}) \approx \sqrt{\frac{p}{2}} \quad \text{(state of the art!)}$$

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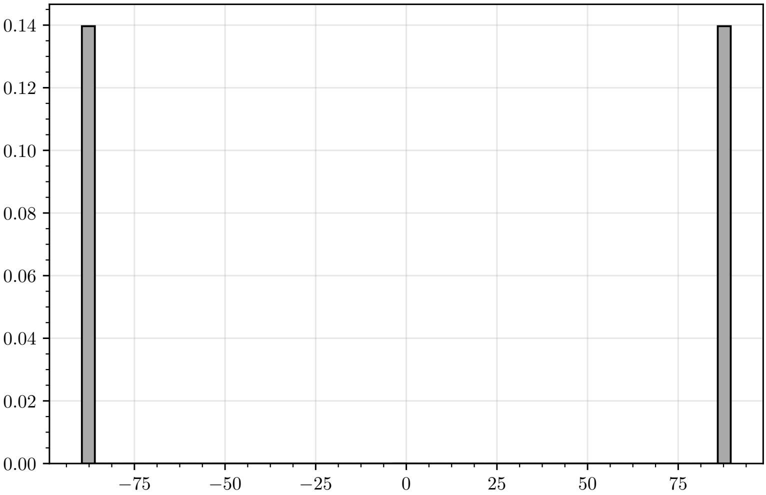
Even simpler is **spectral bound** (Haemers' variation on Hoffman's):

$$\omega(G_p) \le k + \max_{C \text{ a } k\text{-clique in } G_p} f(G_{p,C}),$$

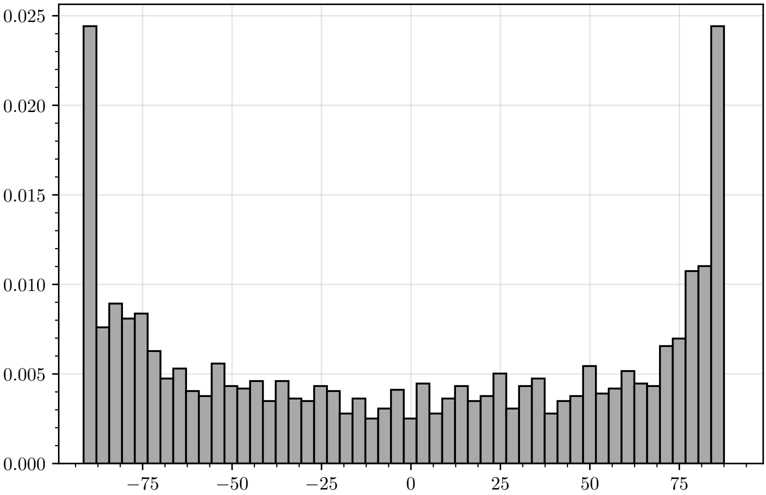
$$f(G) := |V(G)| \left(\frac{\min \deg(\overline{G})^2}{\max \deg(\overline{G}) \cdot |\lambda_{\min}(\overline{G})|} - 1 \right)^{-1}.$$

Main point: Enough to understand **spectrum** of the $G_{p,C}$.

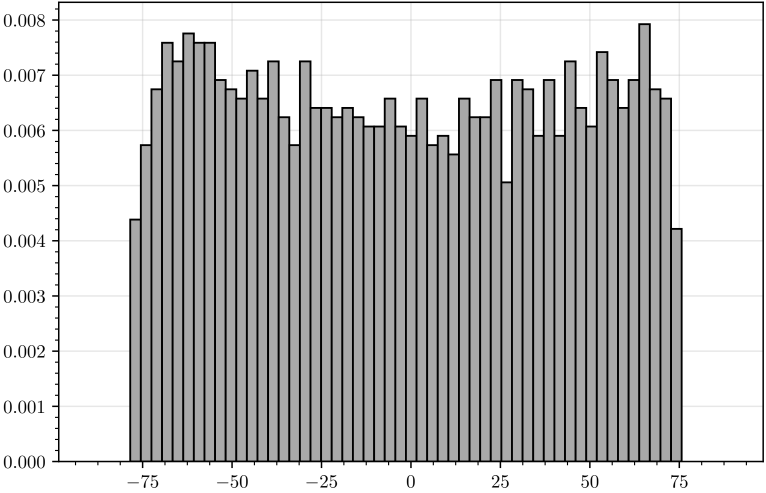
Experiments: $\lambda(G_p)$ ($p \approx 8000$)



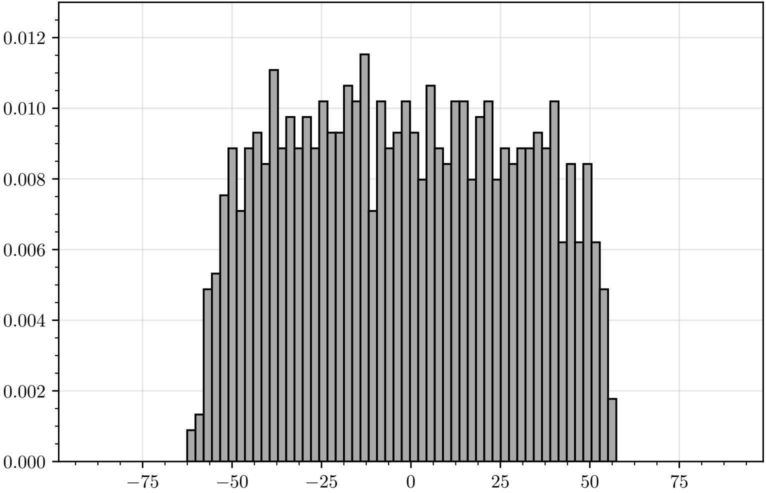
Experiments: $\lambda(G_{p,\{0\}})$



Experiments: $\lambda(G_{p,\{0,1\}})$



Experiments: $\lambda(G_{p,\{0,1,x\}})$



A Probabilist's Old Friend

Definition: The **Kesten-McKay law** with parameter $d \geq 2$ is

$$d\mu_{\mathsf{KM}(d)}(x) = \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)} \mathbb{1}\left\{|x| \leq 2\sqrt{d-1}\right\} dx$$

Also extends to $1 \leq d < 2$ by adding two atoms:

$$d\mu_{\mathsf{KM}(d)}(x) = (\cdots) + \frac{2-d}{2}\delta_{-d}(x) + +\frac{2-d}{2}\delta_d(x).$$

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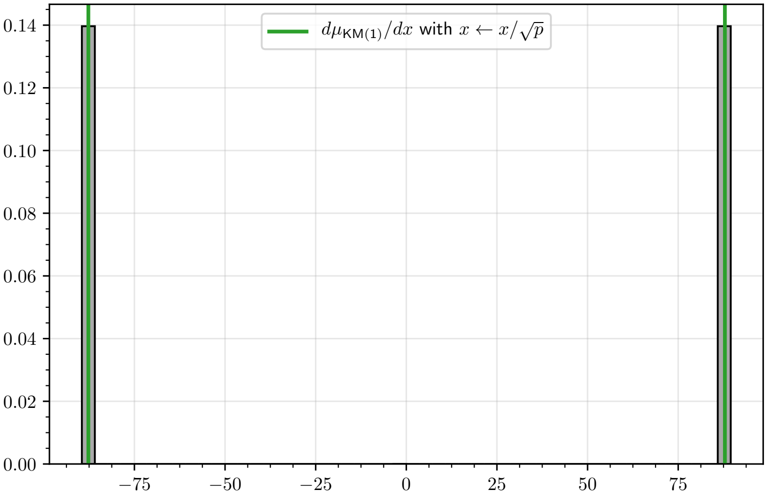
Observation: Up to rescaling and suitable shifting, empirical spectral distribution of $G_{p,C}$ looks like $\mu_{\mathsf{KM}(2^{|C|})}$.

Let's look...

Experiments: $\lambda(G_p)$



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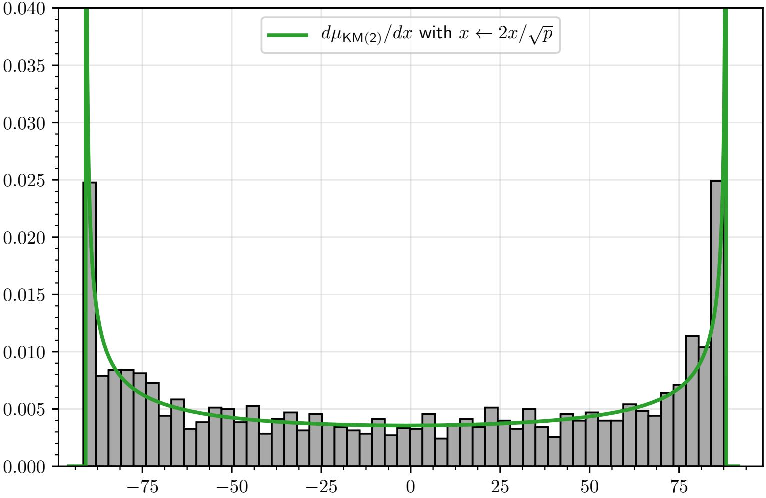

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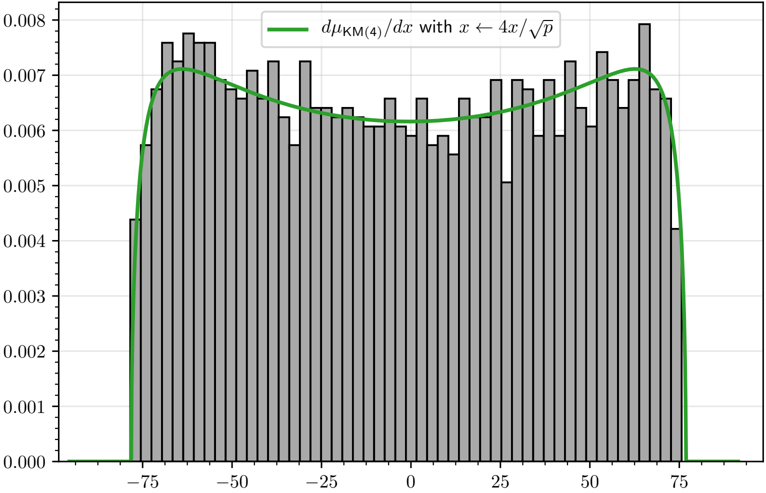

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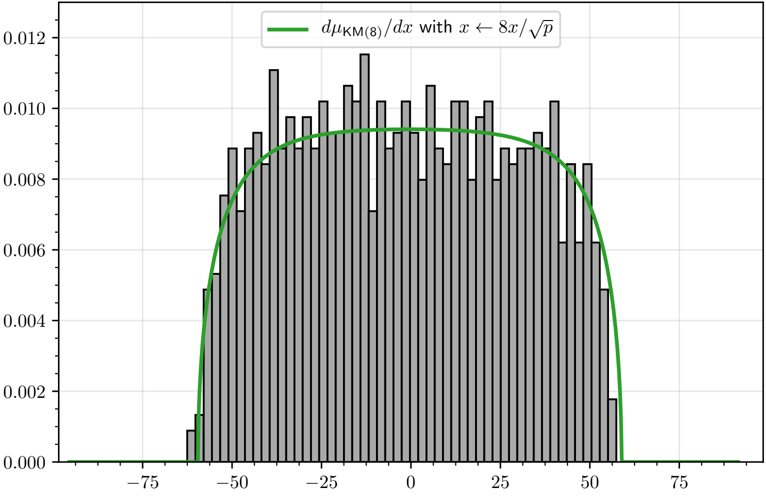

Why Does Kesten-McKay Appear?

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Related to its role in **free probability**:

Theorem: [Voiculescu ’90s] $\boldsymbol{D} \in \mathbb{R}^{N \times N}$ diagonal with $D_{ii} \overset{\mathsf{iid}}{\sim} \mathsf{Unif}(\{\pm 1\})$, $\boldsymbol{U} \sim \mathsf{Haar}(\mathcal{U}(N))$, and $\boldsymbol{M}$ a principal submatrix of $\boldsymbol{UDU}^*$ with each row/column included with probability $\alpha \in (0, 1]$. Then,

$$\text{rescaled empirical spectral distribution of } \boldsymbol{M} \Rightarrow \mu_{\mathsf{KM}(1/\alpha)}.$$

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Idea: **derandomize** this model (in $\boldsymbol{U}, \boldsymbol{D}, \boldsymbol{P}$).

Spectral Pseudorandomness for Local Graphs

Observe that

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Gradual derandomization of asymptotic freeness result:

Reference	Matrix	Intuition
[V '90s]	$\boldsymbol{PUDU^*P}$	
[MMP '19]	$\boldsymbol{P}\boldsymbol{A}_{G_p}\boldsymbol{P}$	pseudorandom eigenspaces
[K '23]	$\boldsymbol{P}_{G_{p,C}}\boldsymbol{A}_{G_p}\boldsymbol{P}_{G_{p,C}}$	pseudorandom vertex set

Precise Statement

Theorem: [K '23] Conditional on a family of natural Legendre symbol character sum estimates, for any sequence $C_p \subset V(G_p)$ of cliques with $|C_p| = k$,

$$\text{rescaled e.s.d. of } \pm 1 \text{ adjacency matrix of } G_{p,C_p} \Rightarrow \mu_{\mathsf{KM}(2^k)}.$$

Can prove estimates for $k = 1$, and make progress for $k = 2$.

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Would imply, for any given constant k,

$$\omega(G_p) \leq k + \frac{\sqrt{2^k - 1}}{2^{k-1}}\sqrt{p} + o(\sqrt{p}) \approx 2^{-k/2}\sqrt{p}.$$

Already $k = 3$ would beat state of the art! And arbitrary k would show $\omega(G_p) = o(\sqrt{p})$, "denting" the $\sqrt{p}$ barrier.

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Theorem: [K '23] In Voiculescu's model, $\boldsymbol{M}$ = random submatrix of $\boldsymbol{UDU}^*$ with inclusion probability α,

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$\rightsquigarrow$ long but plausible road to the case of deterministic $\boldsymbol{M}$.

Open Questions

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3. Proof techniques to analyze **convex relaxations** for matrix models with less and less randomness?

4. What other classical questions can be answered through **pseudorandomness (phenomenon)** leveraged via **convex relaxation (proof technique)**?

Thank you!