# The degree 4 sum-of-squares relaxation of the clique number of Paley graphs


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# Paley Graph

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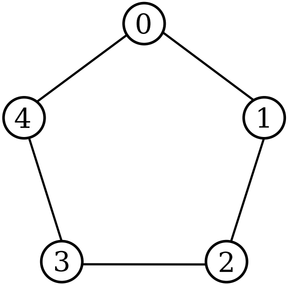
**Example:** $p = 5 \rightsquigarrow$ squares are $\{1, 4 \equiv -1\}$.



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**Heuristic:** $G_p$ is **pseudorandom**, behaving in many ways like $\mathsf{ER}(p, \frac{1}{2})$, i.i.d. random graph with edge probability $\frac{1}{2}$.

**Example:** For any fixed graph $H$, as $p \to \infty$,

$$\begin{aligned} \text{occurrences in } G_p &\sim \mathbb{E}\left[\text{occurrences in } \mathsf{ER}\right] \\ &\sim n^{|V(H)|}\left(\frac{1}{2}\right)^{\binom{|V(H)|}{2}} \end{aligned}$$

# Paley Graphs: The Clique Number

**Question:** What about $H$ growing slowly with $p$?

**Example:** $\omega(G)$ := **largest clique** in $G$. Easy calculations $\Longrightarrow$

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Same for $\omega(G_p)$? Not quite...

$$\omega(G_{p_i}) \geq \log p_i \log\log\log p_i \qquad \text{[Graham, Ringrose '90]}$$

$$\omega(G_p) \overset{?}{\sim} (\log p)^2 \qquad \text{(random heuristic)}$$

And, in any case, the best **upper bounds** we have are

$$\omega(G_p) \leq \sqrt{p} \qquad \text{(spectral/Hoffman/trivial bound)}$$

$$\omega(G_p) \leq \sqrt{p/2} + 1 \qquad \text{[Hanson, Petridis '21]}$$

**Big Question 1: How can we break the "square root barrier" and prove**

$$\omega(G_p) = O(p^{1/2-\varepsilon}) \ ?$$

(Formally similar to controlling the restricted isometry property for the Paley ETF.)

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$$\omega(G) = \max \left\{ \sum_{i \in V} y_i \ : \ \boldsymbol{y} \in \{0,1\}^V, \quad y_i y_j = 0 \text{ if } \{i,j\} \notin E \right\}$$

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3. Optimize $\sum_{i \in V} X(\{i\})$ over that enlarged set.

# Degree 2 $=: \mathsf{SOS}_2(G)$ (Case $d = 1$)

$$\text{maximize} \quad \sum_{i=1}^{p} X(\{i\}) \quad \text{subject to}$$

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$d \geq 2 \rightsquigarrow \mathsf{SOS}_{2d}(G) \geq \omega(G)$, tighter bounds in time $p^{O(d)}$.

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**Question:** How important is the distribution of $\mathsf{ER}(p, \frac{1}{2})$? What properties of a graph does this really depend on?

**Big Question 2: How can we find deterministic graphs $H_p$ with**

$$\omega(H_p) = O(\log p)$$
$$\mathsf{SOS}_{2d}(H_p) = \Omega(p^{1/2}) \text{ ?}$$

# Our Results

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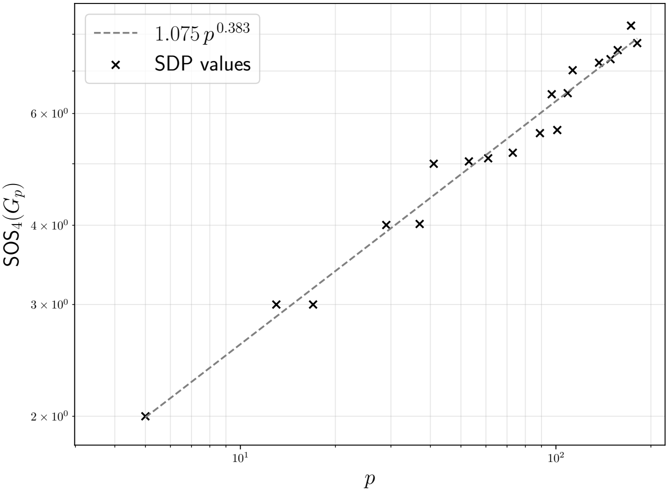
**Main theorem:** [KY] $\mathsf{SOS}_4(G_p) = \Omega(p^{1/3})$.

**Remark:** Derandomizes an early result on the random graph case: [DM '15] showed $\mathbb{E}[\mathsf{SOS}_4(\mathsf{ER})] = \widetilde{\Omega}(p^{1/3})$.

# Ancillary Results I: Improve to $\Omega(p^{1/2})$?




**Exciting observation:** Appear to have $\mathsf{SOS}_4(G_p) \sim p^{0.38...}$.

# Ancillary Results II: Improve to $\Omega(p^{0.38\ldots})$?

We use a simple $\boldsymbol{X}$, first used by [FK '03], later by [MW '13], but ultimately found to be insufficient by [BHKKP '19]:

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**Theorem:** [KY] For Paley graphs, such proves only

$$\mathsf{SOS}_4(G_p) = \Omega(p^{1/3}),$$

i.e., our main result cannot be improved without a fancier choice of $\boldsymbol{X}$ $\rightsquigarrow$ probably significantly harder to analyze.

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Theoretical evidence: [BHKKP '19] proof depends on norm bounds for **graph matrices** formed from the $\{\pm 1\}$ adjacency matrix $\boldsymbol{A}$.

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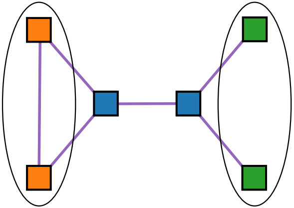
$H =$ 

we get a matrix

$$M^H_{(a,b),(c,d)}(G) \approx \sum_{i,j} A_{a,b} A_{a,i} A_{b,i} A_{i,j} A_{j,c} A_{j,d}.$$

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Basically, can build these by taking advantage of the discrepancy between

$$\boldsymbol{A}_{G_p}^2 = p\boldsymbol{I} - \mathbf{1}\mathbf{1}^\top,$$
$$\boldsymbol{A}_{\mathsf{ER}}^2 = p\boldsymbol{I} + \sqrt{p} \cdot (\text{random matrix}).$$

# Proof Idea

Also boils down to bounding $\|\boldsymbol{M}^H(G_p)\|$ for various $H$, but with different tools.

[AMP '16], [BHKKP '19]: **trace method** using $\mathbb{E}[\mathsf{Tr}((\boldsymbol{M}^H(\mathsf{ER}))^k)]$

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For $\chi : \mathbb{F}_p \to \mathbb{C}$ the **Legendre symbol** character,

$$(\boldsymbol{A}_{G_p})_{i,j} = \left\{ \begin{array}{ll} +1 & \text{if } i \sim j \\ -1 & \text{if } i \not\sim j \end{array} \right\} = \chi(i - j),$$

so polynomials in $\chi$ appear in entries of $\boldsymbol{M}^H$. Not many good tools for handling $\mathsf{Tr}((\boldsymbol{M}^H(G_p))^k)$ character sums, but we can use other case-by-case tricks to avoid these.

# In $\mathbb{F}_p^k$ Nobody Can Hear You Scream

However, in practice it is not always so easy to compute $G_{\text{geom}}$, even when the parameter space is a curve. We often have only meager global information about the sheaf in question, and so we try first to extract and then to exploit information about its local monodromy around each of the points at infinity of the parameter curve. One striking way in which pure lisse sheaves arising from exponential sums differ from the more traditional pure lisse sheaves arising as "cohomology along the fibres, with constant coefficients, of a proper smooth morphism" is that the local monodromy of the former can be quite wildly ramified, and can be so in quite interesting ways. This possibility can often be exploited to impose some very severe restrictions on $G_{\text{geom}}$. The underlying mechanisms of wild ramification and the restrictions it can impose are discussed in Chapter I.

One way in which the invariants and covariants of local monodromy can be detected and analyzed is through their interpretation as the difference between the compactly supported and the ordinary cohomology groups of the parameter curve with coefficients in the sheaf under discussion. This relation, together with a thorough discussion of the basic general facts about curves and their cohomology, is given in Chapter II, and systematically exploited in Chapter VII.

[Katz '88]

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5. How much of the structure of “clique space” of the Paley graph behaves like random graphs?

# Thank you!