Assignment 0

CPSC 663: Sum-of-Squares Optimization (Spring 2022)

This homework will not be graded with a numerical score, and will not be included in the calculation of your final grade. Still, I encourage you to try the problems and, if you are not completely sure that you have solved them correctly, to submit solutions, which I will return with comments to help you assess whether you are prepared for this course. I would expect all enrolled students to be able to solve these problems without too much trouble.

Linear Algebra

Problem 1 (Inner and outer products). Suppose $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathbb{R}^d$. Prove that the matrix

$$\boldsymbol{M} = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle \boldsymbol{v}_i \boldsymbol{v}_j^\top$$

is positive semidefinite. (Here $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = \boldsymbol{v}_i^\top \boldsymbol{v}_j$ is the usual Euclidean inner product.) Try to give two arguments: first consider quadratic forms $\boldsymbol{x}^\top \boldsymbol{M} \boldsymbol{x}$, and second factorize $\boldsymbol{M}$ and use that if $\boldsymbol{M}$ admits a particular type of factorization then $\boldsymbol{M}$ is positive semidefinite.

Problem 2 (Schur product). Suppose $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{d \times d}$ are positive semidefinite. Let $\boldsymbol{M} \in \mathbb{R}^{d \times d}$ have $M_{ij} = A_{ij} B_{ij}$ (note that $\boldsymbol{M} \neq \boldsymbol{AB}$ under ordinary matrix multiplication). Prove that $\boldsymbol{M}$ is also positive semidefinite. Try to find a simple and elegant proof.

Probability

Problem 3 (Moments and Chebyshev). Suppose X is a random variable with $\mathbb{E}[X] = a$ and $\mathbb{E}[X^2] = b$. Chebyshev's inequality states that

$$\mathbb{P}[|X - a| \geq t] \leq \frac{\mathsf{Var}[X]}{t^2} = \frac{b - a^2}{t^2}.$$

(a) Describe exactly the set S of pairs $(a, b) \in \mathbb{R}^2$ for which there exists a random variable X such that $\mathbb{E}[X] = a$ and $\mathbb{E}[X^2] = b$. For $(a, b) \in S$, give an X demonstrating this membership. For $(a, b) \notin S$, prove that no such X can exist.

(b) Show that, when it is not trivial, Chebyshev's inequality is the best possible tail bound depending on the first two moments: for all $(a, b) \in S$ from your solution to Part (a) and for all $t > 0$ such that $(b - a^2)/t^2 \leq 1$, there exists a random variable X with $\mathbb{E}[X] = a$, $\mathbb{E}[X^2] = b$, and

$$\mathbb{P}[|X - a| \geq t] = \frac{b - a^2}{t^2}.$$

Problem 4 (A bit of counting)**.** Suppose we place each of m balls into a uniformly random one of n bins. Show that, if for some $\epsilon > 0$ we have $m \geq (1+\epsilon)n\log n$, then with high probability no bin is empty (that is, if $m = m(n)$ is some function satisfying the lower bound, then $\lim_{n\to\infty} \mathbb{P}[\text{any bin is empty}] = 0$).

Convex Optimization

Problem 5 (LP duality)**.** Consider the following linear program, for some fixed vector $\boldsymbol{v} \in \mathbb{R}^d$:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^d x_i v_i \\ \text{subject to} & -1 \leq x_i \leq 1 \text{ for } 1 \leq i \leq d \end{array}$$

(a) What is a concise description of the optimal value of this (as a function of $\boldsymbol{v}$)? What are the optimal x_i?

(b) Calculate the dual of this program. What are the values of the optimal dual variables?

(c) Describe the consequences of complementary slackness between the primal and dual programs in this case.

My First Sums of Squares

These problems are less about background knowledge for the course (though you should still be able to solve them), and more of a little preview of the kind of restricted "proof system" we will be studying.

Problem 6 (AM-GM)**.** The *arithmetic-geometric mean (AM-GM) inequality* is that $ab \leq \frac{a^2+b^2}{2}$ for $a, b \in \mathbb{R}$ (you might have instead seen the equivalent $\sqrt{cd} \leq \frac{c+d}{2}$ for $c, d > 0$). Give a proof of this by coming up with some polynomial $P(a,b)$ with real coefficients and using that $P(a,b)^2 \geq 0$.

Problem 7 (Cauchy-Schwarz)**.** The *Cauchy-Schwarz inequality* says that, for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$, $\langle \boldsymbol{u}, \boldsymbol{v}\rangle^2 \leq \|\boldsymbol{u}\|^2 \cdot \|\boldsymbol{v}\|^2$, or, written out,

$$\left(\sum_{i=1}^d u_i v_i\right)^2 \leq \left(\sum_{i=1}^d u_i^2\right)\left(\sum_{i=1}^d v_i^2\right).$$

(You might more often see this with square roots taken on either side.) Give a proof of this by coming up with some rational functions $R_i(\boldsymbol{u}, \boldsymbol{v})$ and using that $\sum_i R_i(\boldsymbol{u}, \boldsymbol{v})^2 \geq 0$. (It's a sum of squares!)

> HINT: First, remind yourself of the usual proof of the Cauchy-Schwarz inequality you've once seen. Then, try showing that $\langle \boldsymbol{u}, \boldsymbol{v}\rangle \leq \|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\|$ in this way (note that there is no absolute value on the left), with $R_i(\boldsymbol{u}, \boldsymbol{v})$ being arbitrary real-valued functions. Finally, try adjusting this to prove the stated result with $R_i(\boldsymbol{u}, \boldsymbol{v})$ being rational functions.