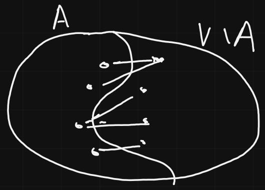
Lecture 1: Invitation + Max Cut

Max cut: $G = (V, E)$

Cut: $A \subseteq V$, size $= \#\{\{v,w\} \in E : v \in A, w \notin A\}$



$\mathrm{MaxCut}(G)$ = largest size of cut.

Df: Graph Laplacian: $\underbrace{\frac{1}{4}(D - A)}_{=:\,L} \in \mathbb{R}^{V \times V}$

diagonal $D_{ii} = \deg(i)$; adjacency $A_{ij} = \mathbb{1}\{\{i,j\} \in E\}$

$$x^T L x = \sum_{\{i,j\} \in E} \left(\frac{x_i - x_j}{2}\right)^2$$

Prop: $\mathrm{MaxCut}(G) = \max_{x \in \{\pm 1\}^V} x^T L x$ $\quad \left(\frac{x_i - x_j}{2}\right)^2 = \begin{cases} 0 & x_i = x_j \\ 1 & x_i \neq x_j \end{cases}$ $\quad (A = \{i : x_i = +1\})$

$\Rightarrow$ MaxCut is case of polynomial opt. over hypercube.

Thm (Karp): MaxCut is NP-complete.

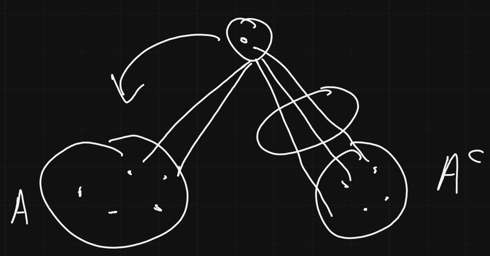
Df: $\hat{x} = \hat{x}(G)$ is an α-approx.[1] if, $\forall G$, $\hat{x}^T L \hat{x} \geq \alpha \max x^T L x = \alpha\, \mathrm{MaxCut}(G)$

Allow $\hat{x}$ random, same thing if $\mathbb{E}\, \hat{x}^T L \hat{x} \geq (-\text{''}-)$.

Prop: $\exists$ a randomized poly time $\frac{1}{2}$-approx.

Pf: $\hat{x}_i \sim \mathrm{Unif}(\{+1, -1\})$ independently.

$\rightsquigarrow \mathbb{E}\, \hat{x}^T L \hat{x} = \frac{1}{2}|E| \geq \frac{1}{2} \mathrm{MaxCut}(G)$



Prop: $\exists$ a deterministic $\frac{1}{2}$-approx.

Q: $\exists$? $\alpha = \left(\frac{1}{2} + \varepsilon\right)$-approx.[1]?

Idea 1: Better iterative alg.? $\rightarrow \frac{1}{2} + O\left(\frac{1}{\text{max deg.}}\right)$, etc.

Idea 2: LP relaxation.

LOGISTICS

- ✓ Zoom + Friday
- ✓ Notes
- ✓ Assignment 0
- ✓ Survey (!)
- ✓ Grades + Project

$$\max_{x\in\{\pm1\}^V} x^T L x = \max_x \langle L, xx^T\rangle$$

$$\langle A,B\rangle = \mathrm{Tr}(A^T B) = \sum_{ij} A_{ij}B_{ij}$$

$$LP(G) := \left\{ \begin{array}{l} \le \max \langle L, X\rangle \\ \text{s.t. } X\in\mathbb{R}^{V\times V}, \text{sym.} \\ X_{ii} = 1 \\ -1 \le X_{ij} \le 1 \\ X_{ij} + X_{jk} + X_{ik} \ge -1 \end{array}\right. \rightsquigarrow O(|V|^3)$$

"$X = xx^T$, $X_{ij} = x_i x_j$ for $x \in \{\pm1\}^V$"

Observations:

① Sometimes "tight": $X^* = \underbrace{x^*}_{\text{optimal for MaxCut}} x^{*T}$ (e.g. planar)

② Sometimes $LP(G) \le (1+\delta)\,\mathrm{MaxCut}(G)$ (e.g. dense ER random graphs)

Goemans-Williamson (GW) Approx.

$$LP(G) \longrightarrow SDP(G) = \begin{array}{l}\max \langle L, X\rangle \\ \text{subj. to } X \succeq 0 \\ \qquad X_{ii} = 1\end{array}$$

$\langle X, vv^T\rangle$ $v^T X v \ge 0$ $\forall v\in\mathbb{R}^V$ (psd)

Before: $-1 \le X_{ij} \le 1$ ←

$$X \succeq 0 \Rightarrow \begin{array}{cc} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{bmatrix} 1 & X_{ij} \\ X_{ji} = X_{ij} & 1 \end{bmatrix} \end{array}$$

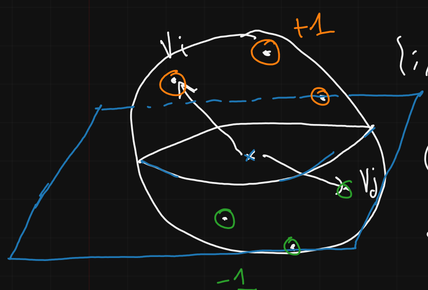
$0 \le \det(\cdots) = 1 - X_{ij}^2$

Rounding: $X \succeq 0 \iff X = V^T V$ for some V

$$V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \rightsquigarrow X_{ij} = \langle v_i, v_j\rangle \quad \text{("Gram matrix")}$$

$X_{ii} = 1 \iff \|v_i\| = 1$

SDP(G) optimizes a vector-valued cut, $\sum L_{ij} \overbrace{\langle v_i, v_j\rangle}^{\text{instead of } x_i x_j}$



$\{i,j\}\in E \rightsquigarrow v_i, v_j$ "repel" on sphere.

$$\underbrace{(v_i)_{i\in V}}_{\text{sphere}} \longmapsto \underbrace{(x_i)_{i\in V}}_{\{\pm1\}}$$

Rk: Can derandomize.

Thm (GW): If X feasible for $SDP(G)$, and $\hat{x} = \hat{x}(X)$ by random hyperplane,

$$\mathbb{E}\langle L, \hat{x}\hat{x}^T\rangle \ge \alpha^{GW}\langle L, X\rangle \qquad \alpha^{GW} = 0.878\ldots$$

Cor: $\exists\ \alpha^{GW}$-approx.

Pf: $X = X^*$, $\mathbb{E}\langle L, \hat{x}\hat{x}^T\rangle \ge \alpha^{GW} SDP(G) \ge \alpha^{GW}\,\mathrm{MaxCut}(G)$.

Duality and SOS:

$$\mathrm{SDP}(G) \overset{\Downarrow}{=} \begin{array}{l} \min \operatorname{Tr}(D) \\ \text{subj. to } D \text{ diagonal} \\ \qquad D \succeq L \quad (\Leftrightarrow D - L \succeq 0) \end{array}$$

$$D - L \succeq 0 \iff D = L + A \text{ for some } \underline{A \succeq 0} \iff A = \sum_a w_a w_a^T = WW^T$$

$$= L + \sum_a w_a w_a^T$$

Aside: H sym. $\rightarrow p_H(y) = y^T H y = \sum_{ij} H_{ij} y_i y_j \qquad H = H' \iff p_H = p_{H'}$

$$\langle w_a, y\rangle^2$$

$$D = L + \sum_a w_a w_a^T \iff \sum \underbrace{D_{ii}}_{d_i} y_i^2 = \sum L_{ij} y_i y_j + \sum_a \Big(\sum_j w_{aj} y_j\Big)^2$$

$$\iff c = \sum L_{ij} y_i y_j + \sum d_i (1 - y_i^2) + \sum_a \Big(\sum_j w_{aj} y_j\Big)^2$$

constant term $\rightarrow c = \sum d_i = \operatorname{Tr}(D)$

Claim:
$$\mathrm{SDP}(G) = \begin{array}{l} \min c \\ \text{subj. to } c \underset{\substack{\uparrow \\ \text{POLY. EQUALITY} \\ \text{for some } (d_i), (w_{aj}).}}{=} \sum L_{ij} y_i y_j + \underbrace{\sum d_i (1 - y_i^2)}_{\substack{=0 \\ \text{at any } y \in \{\pm 1\}^V}} + \overbrace{\underbrace{\sum_a \Big(\sum_j w_{aj} y_j\Big)^2}_{\geq 0}}^{(\mathrm{SOS})} \quad (*) \end{array}$$

$(*)$ is an SOS proof of "$\langle L, xx^T\rangle = \sum L_{ij} x_i x_j \leq c$ whenever $x \in \{\pm 1\}^V$"

i.e.

$\mathrm{MaxCut}(G) \leq c$.

Improvements?

Prob: $\exists ? (\alpha^{GW} + \varepsilon)$-approx^n. Evidence for "no": reduction from UGC.

In particular, "higher-degree" SOS?

$$\begin{array}{l} \min c \\ \text{subj. to } c = \underbrace{\sum L_{ij} y_i y_j} + \sum_i \overbrace{\underbrace{(1 - y_i^2)}}^{\substack{0 \\ \shortparallel}} p_i(y) + \sum s_j(y)^2 \end{array}$$

Coming up: $\deg p_i, \deg s_j \leq D$ const $\rightsquigarrow$ optimize in poly time.