

Lecture 4: SDP implementation of SOS

[Shor '87, Nesterov '00; Lasserre, Parrilo-early '00s]

LOGISTICS:

- ✓ HW schedule
- ✓ Project ideas
- ✓ Switch to in-person

$$\text{OPT} := \begin{cases} \text{maximize} & p(x) \\ \text{subj. to} & x \in \mathbb{R}^n \\ & f_i(x) = 0 \text{ for } i=1, \dots, a \\ & g_j(x) \geq 0 \text{ for } j=1, \dots, b \end{cases}$$

≤ variables
≤ data

Parrilo / "proof" relaxation:

$$\text{Parr}_D := \begin{cases} \text{minimize} & c \\ \text{subj. to} & c - p(x) \stackrel{!}{=} \sum_{i=1}^a f_i(x) q_i(x) + r_0(x) + \sum_{j=1}^b g_j(x) r_j(x) \\ & q_i(x) \in \mathbb{R}[x_1, \dots, x_n] \\ & \deg f_i q_i \leq D \\ & r_0, r_1, \dots, r_j \in \text{SOS} \\ & \deg r_0, \deg g_j r_j \leq D. \end{cases}$$

"degree D SOS proofs"

Def: (Multisets) $\binom{[n]}{k} \stackrel{\subset \text{SET}}{=} \{\text{multisets size } k \text{ in } [n]\} \quad (= S: [n] \rightarrow \mathbb{N}, |S| = \sum_i S(i) = k)$

$\binom{[n]}{\leq k} = \{\text{multisets size between } 0 \text{ and } k\}$. Natural ordering: by size, lex.

$$x^S = \prod_i x_i^{S(i)} = \prod_{i \in S} x_i$$

Def: $x^{\otimes \leq d} = (x^S)_{S \in \binom{[n]}{\leq d}}$ (Ex: $n=2, x^{\otimes \leq 3} =$

$$\begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \\ x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{bmatrix}$$

Def: (Coeff. vector) $p \in \mathbb{R}[x]$ $\deg p \leq D$

$\rightarrow v^{(p,D)} \in \mathbb{R}^{\binom{[n]}{\leq D}}$ vec. of coeff., unique s.t.

$$p(x) \stackrel{!}{=} \langle v^{(p,D)}, x^{\otimes \leq D} \rangle$$

Prop: $p \in \mathbb{R}[x], \deg p \leq D \rightarrow \exists M^{(p,D)}$ s.t. $\forall q$ w/ $\deg pq \leq D$,
 $M^{(p,D)} v^{(q, D-\deg p)} = v^{(pq, D)}$

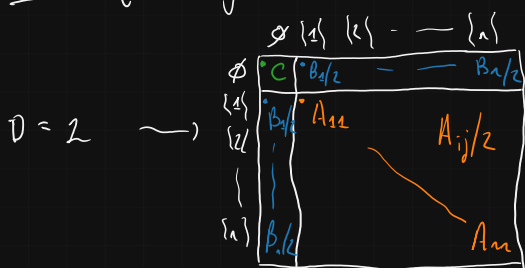
$$c - p(x) = \sum_{i=1}^a f_i(x) q_i(x) + r_0(x) + \sum_{j=1}^b g_j(x) r_j(x)$$

$$\begin{aligned} \underline{v^{(c,D)}} - \underline{v^{(p,D)}} &= \sum_{i=1}^a \underline{M^{(f_i, D)}} \underline{v^{(q_i, D)}} + \sum_{j=1}^b \underline{M^{(g_j, D)}} \underline{v^{(r_j, D)}} \\ &= c e_1 \end{aligned}$$

Now, how to impose $r_j \in \text{SOS}$ on $v^{(r_j, D)}$?

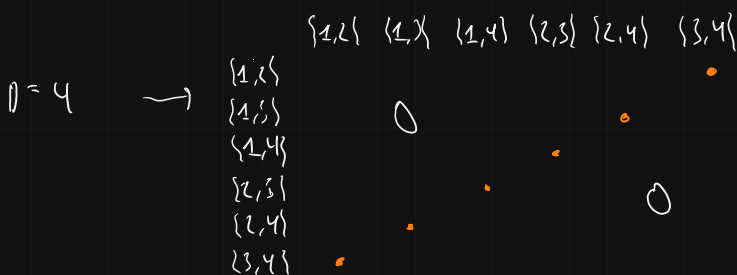
Def: $D \geq 2$ even, $p \in \mathbb{R}[x]$, $\deg p \leq D$. $S \in \mathbb{R}^{\binom{D+1}{\leq D/2} \times \binom{D+1}{\leq D/2}}$ represents p if
 $p(x) \stackrel{(\dagger)}{=} x^{\otimes D/2 T} S x^{\otimes D/2}$

Rk: Not unique in general.



$$x^{\otimes \leq 1} = \begin{bmatrix} 1 \\ x \\ x \\ 1 \end{bmatrix}$$

$$p(x) = \sum_{i,j} A_{ij} x_i x_j + \sum_i B_i x_i + C$$



$$p(x) = x_1 x_2 x_3 x_4$$

S represents p iff $\sum_i \cdot = 1$.

Lem: $p \in \text{SOS} \iff \exists S$ representing p s.t. $S \succeq 0$.

Pf: $p \in \text{SOS} \iff p = \sum_i s_i(x)^2 = \sum_i \left\langle v^{(s_i, D/2)}, x^{\otimes \leq D/2} \right\rangle^2$
 $= x^{\otimes \leq D/2 T} \left(\underbrace{\sum_i v^{(s_i, D/2)} v^{(s_i, D/2) T}}_S \right) x^{\otimes \leq D/2}$

Def: $\text{vec}(S) :=$ columns of S concatenated.

Prop: $D \geq 2$ even $\rightarrow \exists V^{(D)}$ s.t. $\forall S \in \mathbb{R}_{\text{sym}}^{\binom{D+1}{\leq D/2} \times \binom{D+1}{\leq D/2}}$
 $V^{(D)} \text{vec}(S) = v^{(p, D)}$ where S represents p .

Thm: (Perrillo SDP)

$$P_{\text{Perr}_D} = \begin{cases} \text{maximize} & c \\ \text{s.t.} & c e_1 - v^{(p, D)} = \sum_{i=1}^a M^{(f_i, D)} v^{(g_i, D - \deg f_i)} \\ & + V^{(D)} \text{vec}(R_0) \\ & + \sum_{j=1}^b M^{(g_j, D)} V^{(D)} \text{vec}(R_j) \\ & R_0, \dots, R_b \succeq 0 \text{ "satisfiable"} \\ & v^{(g_i, D - \deg f_i)} \in \mathbb{R}^{\binom{D+1}{\leq D - \deg f_i}} \end{cases}$$

SDP!

Observation:

- ① Inequality constr. $g_j(x) \geq 0$ very expensive
- ② Lin. constr. $\rightarrow n^{O(D)}$ indep. of f_i
- ③ $(a+b) n^{O(D)}$ variables, $n^{O(D)}$ constraints, b psd constr.

Dual Lasserre relaxation:

Def: $\tilde{E} : \mathbb{R}[x_1, \dots, x_n]_{\leq D} \rightarrow \mathbb{R}$ is deg. D pseudoexpectation if:

① \tilde{E} linear

② $\tilde{E}[1] = 1$

③ $\tilde{E}[f_i q_j] = 0 \quad \forall i \in [a] \text{ deg } f_i \leq D$

④A $\tilde{E}[s^2] \geq 0$ if $\text{deg } s^2 \leq D$

④B $\tilde{E}[g_j s^2] \geq 0 \quad \forall j \in [b] \text{ deg } g_j s^2 \leq D$.

$\rightarrow m_\emptyset = 1$

$\rightarrow 0 = \langle M^{(f_i, D)} v, m \rangle \quad \forall v$
 $= \langle v, M^{(f_i, D)} \rangle \Leftrightarrow M^{(f_i, D)} \succeq 0$

$\rightarrow \text{mat}(m)_{S, T} = \tilde{E}[x^S x^T]$

④A $\Leftrightarrow \text{mat}(m) \succeq 0$.

$K := \{x : f_i(x) = 0, g_j(x) \geq 0\}$

$$\text{OPT} = \left\{ \begin{array}{l} \max p(x) \\ \text{s.t. } x \in K \end{array} \right\} = \left\{ \begin{array}{l} \max \int p(x) d\mu \\ \text{s.t. } \mu \text{ prob. measure over } K \end{array} \right\} \leq \left\{ \begin{array}{l} \max \tilde{E} p(x) \\ \text{s.t. } \tilde{E} \text{ deg } D \text{ p.e.} \end{array} \right\} =: \text{Lass}_D$$

To implement: Linearity $\rightarrow \tilde{E}$ specified by pseudomoments $m = (\tilde{E} x^S)_{S \in \binom{[n]}{\leq D}} = \tilde{E} x^{\oplus \leq D}$

$\tilde{E} p(x) = \langle v^{(f, D)}, m \rangle$

Thm: $\text{Lass}_D =$ (some SDP).

Prop: Lass_D and Parr_D are dual SDPs.

Thm: (Weak duality) $\text{Lass}_D \leq \text{Parr}_D$

Pf: Feasible pt. for Parr_D :

$$\tilde{E} \left\{ c - p(x) \right\} = \tilde{E} \left[\underbrace{\sum_1^a f_i(x) q_i(x)}_0 + \underbrace{r_0(x)}_{\geq 0} + \sum_1^b \underbrace{g_j(x) r_j(x)}_{\geq 0} \right]$$

Feasible pt. \tilde{E} for Lass_D

$\rightarrow c - \tilde{E} p(x) \geq 0 \rightarrow \tilde{E} p(x) \leq c$.

Thm: (Strong duality) Archimedean (constraints "prox" $\sum_1^r x_i^2 \leq R$) $\Rightarrow \text{Lass}_D = \text{Parr}_D =: \text{SOS}_D$

[Josa, Hermon 108]

Thm: (Convergence) Archimedean $\Rightarrow \lim_{D \rightarrow \infty} \text{SOS}_D = \text{OPT}$.

Pf: $\epsilon > 0$. $\forall x \in K, p(x) < \text{OPT} + \epsilon$. Archimedean + Putinar's Positiv $\Rightarrow \exists$ SOS proof of deg $D(\epsilon)$ of $p(x) \leq \text{OPT} + \epsilon \rightarrow \text{SOS}_D \leq \text{OPT} + \epsilon$.

Thm: (Finite convergence) Often, $\exists D$ st. $\text{SOS}_D = \text{OPT}$.

Wie ~ '10 \rightarrow Archimedean \Rightarrow finite conv. holds "generically" (for "most" problems).