Lecture 4: SDP implementation of SOS

[Shor '87, Nesterov '00; Lasserre, Parrilo - early '00s]

LOGISTICS:

- ✓ HW schedule
- ✓ Project ideas
- ✓ Switch to in-person

$$\mathrm{OPT} := \begin{cases} \text{maximize} & p(x) \\ \text{subj. to} & x \in \mathbb{R}^n \\ & f_i(x) = 0 \quad \text{for } i = 1, \dots, a \\ & g_j(x) \geq 0 \quad \text{for } j = 1, \dots, b \end{cases}$$

Parrilo / "proof" relaxation:

variables

data

$$\mathrm{Parr}_D := \begin{cases} \text{minimize} & c \\ \text{subj. to} & c - p(x) \overset{(p)}{=} \sum_1^a f_i(x) q_i(x) + r_0(x) + \sum_1^b g_j(x) r_j(x) \\ & q_i(x) \in \mathbb{R}[x_1, \dots, x_n] \\ & \deg f_i q_i \leq D \\ & r_0, r_1, \dots, r_j \in \mathrm{SOS} \\ & \deg r_0, \deg g_j r_j \leq D. \end{cases}$$

"degree D SOS proofs"

Def: (Multisets) $\left(\!\!\binom{[n]}{k}\!\!\right)$ = {multisets size k in $[n]$} $\left(= S : [n] \to \mathbb{N},\ |S| = \sum_i S(i) = k\right)$ ← SET

$\left(\!\!\binom{[n]}{\leq k}\!\!\right)$ = {multisets size between 0 and k}. Natural ordering: by size, lex.

$$x^S = \prod_i x_i^{S(i)} = \prod_{i \in S} x_i.$$

Def: $x^{\otimes \leq d} = (x^S)_{S \in \left(\!\binom{[n]}{\leq d}\!\right)}$

Ex: $n = 2$, $x^{\otimes \leq 3} = \begin{bmatrix} 1 \\ \hline x_1 \\ x_2 \\ \hline x_1^2 \\ x_1 x_2 \\ x_2^2 \\ \hline x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{bmatrix}$

Def: (Coeff. vector) $p \in \mathbb{R}[\underline{x}]$, $\deg p \leq D$

$\rightsquigarrow v^{(p,D)} \in \mathbb{R}^{\left(\!\binom{[n]}{\leq D}\!\right)}$ vec. of coeffs, unique s.t.

$$p(x) \overset{(p)}{=} \langle v^{(p,D)}, x^{\otimes \leq D} \rangle.$$

Prop: $p \in \mathbb{R}[\underline{x}]$, $\deg p \leq D \longrightarrow \exists\, M^{(p,D)}$ s.t. $\forall q$ w/ $\deg pq \leq D$,

$$M^{(p,D)} v^{(q, D - \deg p)} = v^{(pq, D)}.$$

$$c - p(x) = \sum_1^a f_i(x) q_i(x) + r_0(x) + \sum_1^b g_j(x) r_j(x)$$

$$\Updownarrow$$

$$v^{(c,D)} - v^{(p,D)} = \sum_1^a M^{(f_i,D)} v^{(q_i,D)} + \sum_1^b M^{(g_j,D)} v^{(r_j,D)}$$

$= c\, e_1$

Now, how to impose $r_j \in \mathrm{SOS}$ on $v^{(r_j,D)}$?

Def: $D \geq 2$ even, $p \in \mathbb{R}[\underline{x}]$, $\deg p \leq D$. $S \in \mathbb{R}_{\text{sym}}^{\binom{[n]}{\leq D/2} \times \binom{[n]}{\leq D/2}}$ represents p if

$$p(x) \overset{(1)}{=} x^{\otimes \leq D/2\, T} S \, x^{\otimes \leq D/2}.$$

Rk: Not unique in general.

$$D = 2 \longrightarrow \begin{array}{c|cccc} & \emptyset & \{1\} & \{2\} \cdots & \{n\} \\ \hline \emptyset & C & B_1/2 & \cdots & B_n/2 \\ \{1\} & B_1/2 & A_{11} & & \\ \{2\} & \vdots & & A_{ij}/2 & \\ \vdots & & & \ddots & \\ \{n\} & B_n/2 & & & A_{nn} \end{array} \qquad x^{\otimes \leq 1} = \begin{bmatrix} 1 \\ | \\ x \\ | \end{bmatrix} \qquad p(x) = \sum_{i \leq j} A_{ij} x_i x_j + \sum_i B_i x_i + C$$

$$D = 4 \longrightarrow \begin{array}{c|cccccc} & \{1,2\} & \{1,3\} & \{1,4\} & \{2,3\} & \{2,4\} & \{3,4\} \\ \hline \{1,2\} & & & & & & \cdot \\ \{1,3\} & & 0 & & & \cdot & \\ \{1,4\} & & & & \cdot & & \\ \{2,3\} & & & \cdot & & & \\ \{2,4\} & & \cdot & & & 0 & \\ \{3,4\} & \cdot & & & & & \end{array} \qquad p(x) = x_1 x_2 x_3 x_4$$

S represents p iff $\sum \cdot = 1$.

Lem: $p \in SOS \iff \exists\, S$ representing p s.t. $S \succeq 0$.

Pf: $p \in SOS \iff p = \sum_i s_i(x)^2 = \sum_i \left\langle v^{(s_i, D/2)}, x^{\otimes \leq D/2} \right\rangle^2$

$$= x^{\otimes \leq D/2\, T} \underbrace{\left(\sum_i v^{(s_i, D/2)} v^{(s_i, D/2)\, T} \right)}_{S} x^{\otimes \leq D/2}$$

Def: $\operatorname{vec}(S) :=$ columns of S concatenated.

Prop: $D \geq 2$ even $\longrightarrow \exists\, V^{(D)}$ s.t. $\forall S \in \mathbb{R}_{\text{sym}}^{\binom{[n]}{\leq D/2} \times \binom{[n]}{\leq D/2}}$

$V^{(D)} \operatorname{vec}(S) = v^{(p,D)}$ where S represents p.

Thm: (Parrilo SDP)

$$\mathrm{Parr}_D = \begin{cases} \text{minimize} & c \\ \text{s.t.} & c e_1 - v^{(p,D)} = \sum_{i=1}^{a} M^{(f_i, D)} v^{(q_i, D - \deg f_i)} \\ & \quad + V^{(D)} \operatorname{vec}(R_0) \\ & \quad + \sum_{j=1}^{b} M^{(g_j, D)} V^{(D)} \operatorname{vec}(R_j) \\ & R_0, \dots, R_b \succeq 0 \text{ "suitable"} \\ & v^{(q_i, D - \deg f_i)} \in \mathbb{R}^{\binom{[n]}{D - \deg f_i}}. \end{cases}$$

SDP!

Observations:

① Inequality constr. $g_j(x) \geq 0$ very expensive

② Lin. constr $\rightsquigarrow n^{O(D)}$ indep. of f_i

③ $(a+b)\, n^{O(D)}$ variables, $n^{O(D)}$ lin. constraints, b psd contr.

Dual Lasserre relaxation:

Def: $\tilde{\mathbb{E}} : \mathbb{R}[x_1, \dots, x_n]_{\le D} \longrightarrow \mathbb{R}$ is deg. D pseudoexpectation if:

① $\tilde{\mathbb{E}}$ linear

② $\tilde{\mathbb{E}}[1] = 1$ $\longrightarrow$ $m_\emptyset = 1$

③ $\tilde{\mathbb{E}}[f_i q_i] = 0$ $\forall i \in [a]$ $\deg f_i q \le D$ $\longrightarrow$ $0 = \langle M^{(f_i, D)} v, m \rangle$ $\forall v$
$= \langle v, M^{(f_i,D)^T} m \rangle \leftrightarrow M^{(f_i,D)^T} m = 0$

④A $\tilde{\mathbb{E}}[s^2] \ge 0$ if $\deg s^2 \le D$ $\longrightarrow$ $\mathrm{mat}(m)_{S,T} = \tilde{\mathbb{E}}[x^S x^T]$

④B $\tilde{\mathbb{E}}[g_j s^2] \ge 0$ $\forall j \in [b]$ $\deg g_j s^2 \le D$.

④A $\leftrightarrow$ $\mathrm{mat}(m) \succeq 0$.

$$K := \{x : f_i(x) = 0,\ g_j(x) \ge 0\}$$

$$\mathrm{OPT} = \left\{\begin{matrix} \max & p(x) \\ \text{s.t.} & x \in K \end{matrix}\right\} = \left\{\begin{matrix} \max & \mathbb{E}_\mu\, p(x) \\ \text{s.t.} & \mu \text{ prob. measure over } K \end{matrix}\right\} \le \boxed{\left\{\begin{matrix} \max & \tilde{\mathbb{E}}\, p(x) \\ \text{s.t.} & \tilde{\mathbb{E}} \text{ deg } D \text{ p.e.} \end{matrix}\right\} =: \mathrm{Lass}_D}$$

To implement: Linearity $\rightsquigarrow$ $\tilde{\mathbb{E}}$ specified by pseudomoments $m = \left(\tilde{\mathbb{E}} x^S\right)_{S \in \binom{[n]}{\le D}} = \tilde{\mathbb{E}}\, x^{\otimes \le D}$

$$\tilde{\mathbb{E}}\, p(x) = \langle v^{(p,D)}, m \rangle$$

Thm: $\mathrm{Lass}_D = $ (some SDP).

Prop: Lass_D and Parr_D are dual SDPs.

Thm: (Weak duality) $\mathrm{Lass}_D \le \mathrm{Parr}_D$

Pf: Feasible pt. for Parr_D:

$$\tilde{\mathbb{E}}\left\{c - p(x)\right\} = \tilde{\mathbb{E}}\left\{\sum_1^a \underbrace{f_i(x) q_i(x)}_{0} + \underbrace{r_0(x)}_{\ge 0} + \sum_1^b \underbrace{g_j(x) r_j(x)}_{\ge 0}\right\}$$

Feasible pt. $\tilde{\mathbb{E}}$ for Lass_D

$\rightsquigarrow c - \tilde{\mathbb{E}}\, p(x) \ge 0 \rightsquigarrow \tilde{\mathbb{E}}\, p(x) \le c$.

Thm: (Strong duality) Archimedean (constraints "prove" $\sum_i x_i^2 \le R$) $\Rightarrow$ $\underbrace{\mathrm{Lass}_D = \mathrm{Parr}_D}_{=: \mathrm{SOS}_D}$.

[Josz, Henrion '08]

Thm: (Convergence) Archimedean $\Longrightarrow$ $\lim_{D \to \infty} \mathrm{SOS}_D = \mathrm{OPT}$.

Pf: $\varepsilon > 0$. $\forall x \in K$, $p(x) < \mathrm{OPT} + \varepsilon$. Archimedean + Putinar Psatz $\Rightarrow$ $\exists$ SOS proof of deg $D(\varepsilon)$ of $p(x) \le \mathrm{OPT} + \varepsilon$ $\longrightarrow$ $\mathrm{OPT} \le \mathrm{SOS}_D \le \mathrm{OPT} + \varepsilon$.

Rk: (Finite convergence) Often, $\exists D$ s.t. $\mathrm{SOS}_D = \mathrm{OPT}$.

Nie ~'10 $\rightsquigarrow$ Archimedean $\Rightarrow$ finite conv. holds "generically" (for "most" problems).