Assignment 1

CPSC 664 (Spring 2023)

Modern Probability for Theoretical Computer Science

Assigned: February 15, 2023 Due: March 10, 2023

Solve any three out of the five problems. If you solve them all, I will grade the first three. Each problem will be worth an equal amount towards your grade.

Problem 1 (Coupon collector problem)**.** Suppose that, every day, you receive one of n possible coupons in the mail, chosen uniformly at random (with replacement between different days). We want to know: how many days will it take for you to collect at least one copy of each coupon? Suppose this process goes on for m days.

1. Use the first moment method to show that, if $m = m(n) \geq (1+\varepsilon)n\log n$, then with high probability as $n \to \infty$ after m days you will have a copy of each coupon.

2. Use the second moment method to show that, if $m = m(n) \leq (1-\varepsilon)n\log n$, then with high probability as $n \to \infty$ after m days you will *not* have a copy of some coupon.

Here $\varepsilon > 0$ is a constant not depending on n. The way the problem is formulated above should suggest to you the "right" choice of the counting random variable Z to analyze.

Problem 2 (Sherrington-Kirkpatrick Hamiltonian)**.** Let $\boldsymbol{W}$ be drawn from the unnormalized Gaussian orthogonal ensemble, i.e., having $W_{ij} = W_{ji} \sim \mathcal{N}(0,1)$ and $W_{ii} \sim \mathcal{N}(0,2)$ independently. Consider the random optimization problem

$$\mathsf{opt}(\boldsymbol{W}) := \max_{\boldsymbol{x} \in \{\pm 1\}^n} |\boldsymbol{x}^\top \boldsymbol{W} \boldsymbol{x}|. \tag{1}$$

1. Use the first moment method to prove an upper bound of the form "$\mathsf{opt}(\boldsymbol{W}) \leq Cn^K$ with high probability as $n \to \infty$." Make K as small as possible. You may look up and cite a suitable tail bound on a Gaussian random variable.

2. Show a lower bound of matching order (i.e., a lower bound of the form "$\mathsf{opt}(\boldsymbol{W}) \geq cn^K$ with high probability as $n \to \infty$" for a different K and some $0 < c \leq C$) by the following argument. Let $\boldsymbol{v}$ be a unit eigenvector of $\boldsymbol{W}$ having eigenvalue $\|\boldsymbol{W}\|$. Let $\boldsymbol{x} \in \{\pm 1\}^n$ have $x_i = \mathsf{sgn}(v_i)$. By expanding $\boldsymbol{x}$ in the basis of eigenvectors of $\boldsymbol{W}$, argue that, with high probability, $|\boldsymbol{x}^\top \boldsymbol{W} \boldsymbol{x}| \geq \delta |\boldsymbol{v}^\top \boldsymbol{W} \boldsymbol{v}| \cdot n = \delta \|\boldsymbol{W}\| \cdot n$ for some small constant $\delta > 0$. You may use that $\boldsymbol{W}$ is rotationally invariant, meaning that it has the same law as $\boldsymbol{Q}\boldsymbol{W}\boldsymbol{Q}^\top$ for any orthogonal matrix $\boldsymbol{Q}$.

Problem 3 (Half-and-half SAT[1]). Consider a variant of $2k$-SAT where we view a clause as satisfied if exactly k of its literals are true and k of its literals are false. Call such a problem $2k$-HALFSAT. Consider a random $2k$-HALFSAT instance on n variables and $m = \alpha n$ randomly chosen clauses, as in the SAT and NAESAT models from class.

1. Use the first moment method to find an $\alpha_1 = \alpha_1(2k)$ such that, when $\alpha > \alpha_1$, then a random $2k$-HALFSAT formula is unsatisfiable with high probability. Give the asymptotics of $\alpha_1(2k)$ as $k \to \infty$.

2. Suppose $2k = 4$. Adapt the second moment method from class (carefully up to the heuristics we allowed ourselves in class) to describe an $\alpha_2 = \alpha_2(4) < \alpha_1(4)$ such that, when $\alpha < \alpha_2$, then a random $2k$-HALFSAT formula is satisfiable with high probability. Your description should be in terms of the critical points of an explicit univariate function. Estimate the optimal α_2 by plotting the relevant "overlap curves."

3. Make a conjecture about whether or not

$$\alpha_1(2k) \stackrel{?}{=} (1 + o(1))\alpha_2(2k), \tag{2}$$

as $k \to \infty$, i.e., whether the "vanilla" first and second moments identify the critical α to leading order, as Achlioptas and Moore showed for NAESAT. Support your conjecture with numerical experiments (i.e., plots of overlap curves for larger k).

Problem 4 (Random entire functions). Consider the following random power series: let $a_k \sim \mathcal{N}(0, \frac{1}{k!})$ for $k \geq 0$ be independent Gaussian random variables. Write

$$f(z) := \sum_{k \geq 0} a_k z^k. \tag{3}$$

1. Show that, with probability 1, this power series has infinite radius of convergence. (You may look up and use Gaussian tail bounds and the Borel-Cantelli lemma.) Thus, with probability 1, this random f is an *entire* function (holomorphic on all of $\mathbb{C}$).

2. Is this Gaussian process stationary? You may assume here and below that $f(z)$ is a Gaussian process that, even though it is an infinite Gaussian series for any fixed z, may be manipulated formally in the same way as the finite Gaussian sums from class.

3. You may assume that the Kac-Rice formula for counting zeros of functions of one variable applies to $f(z)$. Use the Kac-Rice formula to compute the expected number of zeros of f in any interval $[a, b] \subset \mathbb{R}$. (Remember our discussion of using the covariance kernel $K(x, y) := \mathbb{E}[f(x)f(y)]$ to compute the covariance of $f(x)$ and $f'(x)$ for any particular x. Also, you may look up how to do Gaussian conditioning calculations if you are not familiar with them.) In words, what does the answer lead you to expect about the distribution of the real zeros of f?

[1] This is Problem 14.30 from The Nature of Computation.

Problem 5 (Local statistics of isotropic fields)**.** Let $f : \mathbb{R}^N \to \mathbb{R}$ be a random Gaussian function, i.e., having $(f(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{R}^N}$ form a real-valued Gaussian process. Suppose $\mathbb{E}f(\boldsymbol{x}) = 0$ for all $\boldsymbol{x}$, that with probability 1 f is differentiable arbitrarily many times (i.e., is C^∞), and that the covariance kernel is of the form

$$K(\boldsymbol{x}, \boldsymbol{y}) = \mathbb{E}[f(\boldsymbol{x})f(\boldsymbol{y})] = k\left(\frac{1}{2}\|\boldsymbol{x} - \boldsymbol{y}\|^2\right), \tag{4}$$

for $\|\cdot\|$ the usual Euclidean norm and $k : \mathbb{R} \to \mathbb{R}$ some smooth function. This is a special case of the concept of a stationary process that we saw in class, sometimes called an *isotropic* process.

1. Let $a, b \in \mathbb{N}$ have different parity (one even and one odd), and let $i_1, \dots, i_a, j_1, \dots, j_b \in [N]$. Show that
$$\mathbb{E}\left[\frac{\partial^a f}{\partial x_{i_1}\cdots\partial x_{i_a}}(\boldsymbol{x})\,\frac{\partial^b f}{\partial x_{j_1}\cdots\partial x_{j_b}}(\boldsymbol{x})\right] = 0. \tag{5}$$
Give a fully rigorous proof without heuristics steps (except that you may exchange differentiation and expectations without justification). Conclude that, for any $\boldsymbol{x}$, the gradient $\nabla f(\boldsymbol{x})$ is independent of both the value $f(\boldsymbol{x})$ and the Hessian $\nabla^2 f(\boldsymbol{x})$.

2. For any given $\boldsymbol{x}$, write down the joint covariance matrix of $(f(\boldsymbol{x}), \nabla f(\boldsymbol{x}), \nabla^2 f(\boldsymbol{x}))$ in terms of the function k (and its derivatives).