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# Lecture 8: Continuous Counting with the Kac-Rice Formula

## 1 Problem Statement

So far in the class, we've applied moment methods to a random variable of interest $Z$ which counts discrete things, such as $Z \coloneqq \#\{x \text{ satisfying } F\}$ for the random $k$-SAT problem. Here, $F : \{0,1\}^n \to \{0,1\}$ was the random Boolean formula acting on $n$ Boolean input values. Today, we're turning our attention to continuous landscapes and solving counting problems over a continuous set.

- **Given:** A random function $F : \mathbb{R} \to \mathbb{R}$.
- **Goal:** For $N = N(F, A) \coloneqq \#\{x \in A \subseteq \mathbb{R} : F(x) = 0\}$, calculate $\mathbb{E}N$, $\mathbb{E}N^2$.

Examples of such a function $F$ include:

- $F(x) = \sum_{a=0}^{d} g_a x^a$, where each $g_a \sim \mathcal{N}(0, \sigma_a^2)$ (Kac)
- $F(x) = \sum_{a=0}^{d} g_a \,\mathsf{cos}(ax)$, again each $g_a \sim \mathcal{N}(0, \sigma_a^2)$ (Rice)

If we think of $A$ as some interval in $\mathbb{R}$, like $A = [a, b]$, we can think of $N$ as counting the number of roots of, say, a random polynomial in this interval. However, $A$ is some uncountable set (for instance, not something like the unit hypercube $\{0,1\}^n$), so we cannot write $N$ in the form

$$N \neq \sum_{x \in A} \mathbb{1}\{F(x) = 0\}.$$

Instead, we need to express it in integral form

$$N = \int_A (\;\cdots\;)\, dx.$$

We will use the Dirac delta function $\delta(\cdot)$, which is characterized by the formal property "$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$."

**Aside:** The Dirac delta function is not a function in the traditional sense. We can define the Dirac measure which accepts a subset $A$ of the real line as an argument and returns $\delta(A) = 1$ if $0 \in A$ and $\delta(A) = 0$ otherwise. Then the Lebesgue integral with respect to $\delta$ satisfies

$$\int_{-\infty}^{\infty} f(x)\delta(dx) = f(0).$$

The measure $\delta$ is not absolutely continuous with respect to the Lebesgue measure, so no Radon-Nikodym derivative with respect to Lebesgue measure (that is, no density) exists, and so we cannot actually write

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0),$$

but if we understand this expression as the former, the latter is a slight abuse of notation.

# 2 Deriving the Formula

## 2.1 One Dimension

**Intuitive motivation.** While considering how to write what $N$ is, we also want to consider invariances in the problem (e.g., we do not want the formula to depend on stretching the interval). For this, recall that we can represent a usual integral which is the area over a flat surface, as a line integral which is the area over a curved surface.

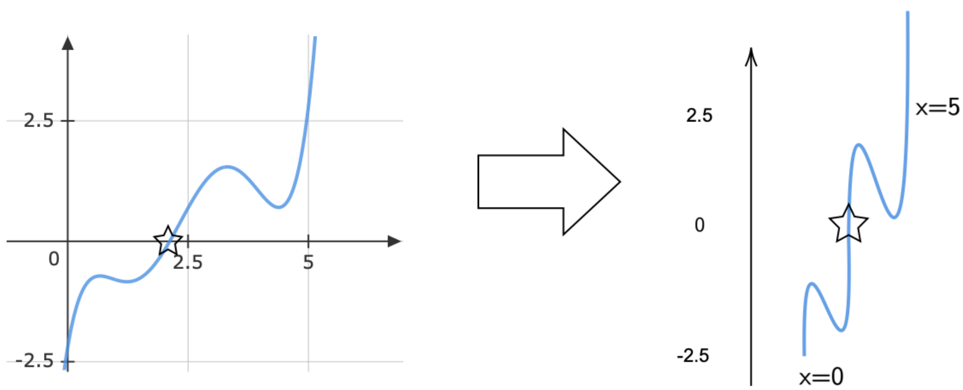


Figure 1: (Left) Integral of a function over $x$ between $x = 0$ and $x = 5$ can be represented as a line integral over the curve $\gamma$ (Right).

Define

$$N \stackrel{\star}{=} \int_{\gamma} \delta(\omega)d\omega = \int_a^b \delta(F(x)) \underbrace{|F'(x)|}_{\text{“length”}} dx.$$

Then we have

$$\mathbb{E}N = \int_a^b \mathbb{E}[\underbrace{\delta(F(x))\,|F'(x)|}_{\text{r.v. parameterized by } x}]dx.$$

Conditioning on $F(x) = y$:

$$\begin{aligned}
\mathbb{E}N &= \int_a^b \int_{-\infty}^{\infty} \mathbb{E}\left[\delta(F(x))\,|F'(x)| \;\middle|\; F(x) = y\right] p_{F(x)}(y)\,dy\,dx \qquad (p_{F(x)} \text{ is density of } F(x)) \\
&= \int_a^b \int_{-\infty}^{\infty} \delta(F(y))\mathbb{E}\left[|F'(x)| \;\middle|\; F(x) = y\right] p_{F(x)}(y)\,dy\,dx \\
&= \boxed{\int_a^b \mathbb{E}\left[|F'(x)| \;\middle|\; F(x) = 0\right] p_{F(x)}(0)dx} \qquad \text{(1D Kac-Rice Formula)}
\end{aligned}$$

**More carefully (Kac counting formula).** Suppose $F(x)$ satisfies the following conditions:

- $F \in \mathcal{C}^1$, that is $F$ continuously differentiable.
- $F(a) \neq 0$ and $F(b) \neq 0$. (Otherwise, the contribution around $F(a)$ or $F(b)$ would be $\epsilon$, and thus contribute $1/2$ to $N$ which ideally should be a whole number.)
- $F(x) = 0 \Rightarrow F'(x) \neq 0$. That is, we need the root to be a "full crossing," e.g., not a flat landscape.

Then, the *Kac counting formula* says that, for all $\epsilon > 0$ sufficiently small, we have

$$N = \frac{1}{2\epsilon} \int_a^b \mathbb{1}_{[-\epsilon,\epsilon]}(F(x))\,|F'(x)|\,dx.$$

See Figure 2.

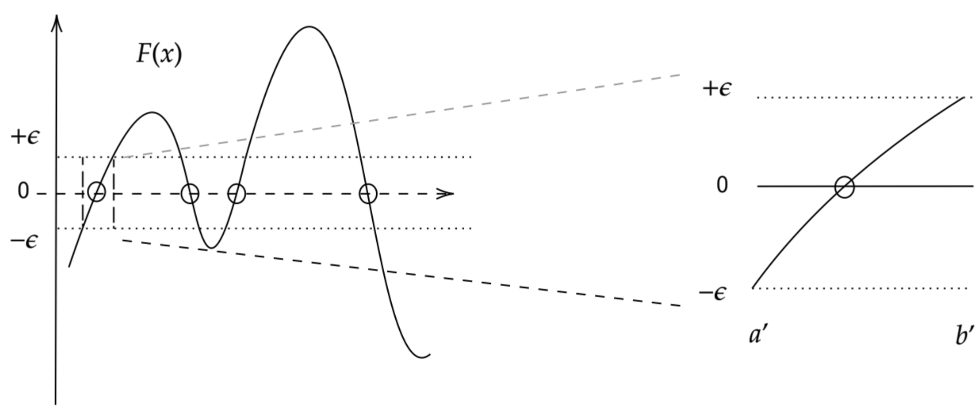


Figure 2: As long as the function $F(x)$ is "nicely behaved," the contribution of each root is, by FTC, $\int_{a'}^{b'} |F'(x)|\,dx = 2\epsilon$. Thus, it contributes 1 to $N$.

## 2.2 Multiple Dimensions

Generalizing the 1-dimensional formula, suppose instead that $F$ is a function

$$F : \mathbb{R}^N \to \mathbb{R}^N.$$

In this case, zeros of $F$ are "generically isolated," so it still makes sense to count them. In addition, the following changes occur:

- The set $A$ is some measurable set in $\mathbb{R}^N$.
- The curve $\gamma$ becomes a manifold embedded in $\mathbb{R}^N$.
- The curve length $|F'(x)|$ becomes the determinants of the Jacobian of $F$, i.e., $|\mathsf{det}\,((DF)(x))|$ which can be seen as infinitesimal area or $N$-volume.

Then our $N$ becomes now

$$N = \int_\gamma \delta(\omega) d\omega = \int_A \delta(F(x))\, |\mathsf{det}\,((DF)(x))|\, dx.$$

And subsequently,

$$\mathbb{E}N = \int_A \mathbb{E}\left[ |\mathsf{det}\,((DF)(x))| \,\middle|\, F(x) = 0 \right] p_{F(x)}(0)\, dx \qquad (N\text{-dim Kac-Rice formula})$$

Note that 0 here is the $N$-dimensional 0 vector.

# 3 Applications

One important application of the Kac-Rice formula is to counting the critical points of a random function.

- **Given:** A function $f : \mathbb{R}^N \mapsto \mathbb{R}$. Define $F(x) := \nabla f(x)$. Note that $F : \mathbb{R}^N \mapsto \mathbb{R}^N$.
- **Quantity of interest:** $N(f, A) = \#\{\text{critical points of } f \text{ in } A\} := \text{Crit}(f, A)$.

In this case, the Jacobian is the Hessian $(DF)(x) = \nabla^2 f(x)$, so the $N$-dimensional Kac-Rice formula yields the following:

$$\mathbb{E}\text{Crit}(f, A) = \int_A \mathbb{E}\left[ \left|\mathsf{det}\left(\nabla^2 f(x)\right)\right| \,\middle|\, \nabla f(x) = 0 \right] p_{\nabla f(x)}(0)\, dx.$$

This is the Kac-Rice formula for critical points.

**Remarks.** We often want to find critical points subject to some further constraints, such as:

- Constraints on the value of $f(x)$, like $f(x) \in B \subseteq \mathbb{R}$.
- Specific "index" of the critical point, e.g., "saddleness" of the critical point (the number of positive eigenvalues of the Hessian).

For these, we can add an indicator in the formula for $N$ under the integral, and the same calculations will go through. We can also in principle write similar integral formulas for the higher moments $\mathbb{E}\text{Crit}(f, A)^k$ is, but in practice this is very hard to compute except in special very symmetric settings.

**Example.** Let

$$f(x) = \frac{\alpha}{2}\|x\|^2 + g(x),$$

where

$$g(x) = \sum_{\substack{i_1,\ldots,i_d \in [N] \\ s_1,\ldots,s_d \in \{0,1\}}} \underbrace{w_{i_1,\ldots,i_d,s_1,\ldots,s_d}}_{\sim\mathcal{N}(0,1)} \begin{pmatrix} \cos(x_{i_1}) & \text{if } s_1 = 0 \\ \sin(x_{i_1}) & \text{if } s_1 = 1 \end{pmatrix} \cdots \begin{pmatrix} \cos(x_{i_d}) & \text{if } s_d = 0 \\ \sin(x_{i_d}) & \text{if } s_d = 1 \end{pmatrix}.$$

Note that $g(x)$ is a Gaussian process with $\mathbb{E}g(x) = 0$. If we define

$$K(x,y) \coloneqq \mathbb{E}g(x)g(y) = \left(\cos(x_1 - y_1) \cdots + \cos(x_N - y_N)\right)^d,$$

then this is also equal to $K(x-y)$, which says that the process is "stationary," which will be explored more in the next lecture.

Intuitively, $f(x)$ looks like a paraboloid plus some "spatially uniform" noise, illustrated below. Asking whether this $f$ has critical points is asking whether the well is "steep" enough, so we can overcome the noise to reach 0 by a process like gradient descent.

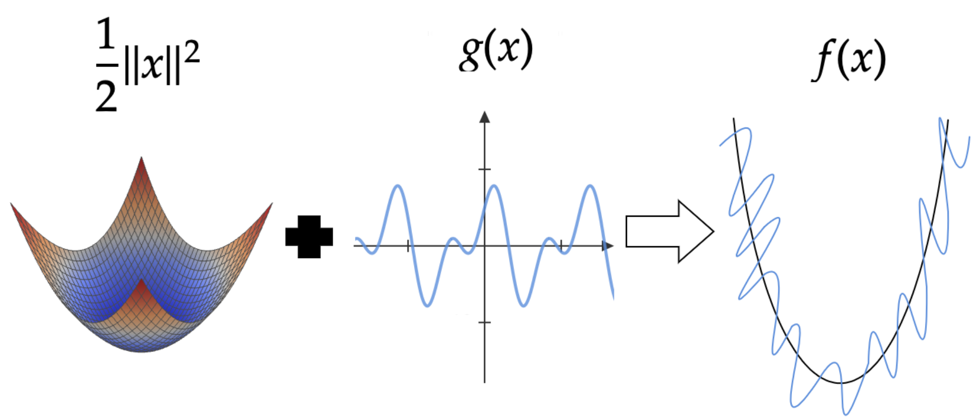


Figure 3: Intuitive illustration of what $f(x)$ looks like.

**Next time.** To complete our calculation, we will calculate correlations between the derivatives of $g$ in terms of the covariance by differentiating under the expectation. This will give identities that look like:

$$\mathbb{E}\left[\frac{\partial g}{\partial x_i}(x)\frac{\partial g}{\partial x_j \partial x_k}(x)\right] = \frac{\partial^3 K}{\partial x_i \partial y_j, \partial y_k}(x,x).$$

We will find that, for any fixed $x$, $\nabla f(x)$ and $\nabla^2 f(x)$ are independent. This means that the conditioning in the Kac-Rice formula has no effect:

$$\mathbb{E}\mathrm{Crit}(f,A) = \int_A \mathbb{E}\left[\left|\det \nabla^2 f(x)\right| \mid \cancel{\nabla f(x) = 0}\right] p_{\nabla f(x)}(0)dx.$$

Most of our work will then be the analysis of the random matrix $\nabla^2 f(x)$.