Lecture 23: Cavity Method I

RS regime of SK model (μ_β, β small). Gibbs measure $\mu(\beta) \propto \mathsf{exp}(\beta x^\intercal W x)$. Idea: "reverse induction" on N ("cavity" left from removing x_i). At size $N+1$ $x = (x_0, \ldots, x_N)$. Let $\hat{x} = (x_1, \ldots, x_N)$ and $\hat{W}$ be the $N \times N$ bottom-right submatrix of W. Say $W_{ii} = 0$. Then

$$x^\intercal W x = \hat{x}^\intercal \hat{W} \hat{x} + x_0 \sum_1^N W_{0i} := \hat{x}^\intercal \hat{W} \hat{x} + h(x_1, \ldots, x_N)$$

We say h is the "cavity field," the effect x_0 feels from the other x_i.

$$\begin{aligned} \Pr_{(x_0,\ldots,x_N)\sim\mu_{\beta,N+1}}[x_0 = s\hat{h}(x_1,\ldots,x_N) = h] &\propto \sum_{\substack{x:x_0=s,\\ h(x_1,\ldots,x_N)=s}} \mathsf{exp}(\beta x^\intercal W x) \\ &= e^{\beta s h} \Pr_{(x_1,\ldots,x_N)\sim\mu_{\beta,N}}[h(x_1,\ldots,x_N) = h] \end{aligned}$$

With the replica method, we began taking expectations over W immediately, whereas with the cavity method we wait. The next question we may ask is what is the distribution of $h(x_1, \ldots, x_N)$ under $\mu_{\beta,N}$? We will answer this by:

1. Conditioning on which cluster $x \sim \mu_{\beta,N}$ lands in.

$$\langle f(x) \rangle_N = \mathbb{E}_{\substack{x\sim\mu_{\beta,N},\\ \text{cond. on one cluster}}} f(x)$$

2. Assume under $\langle \cdot \rangle_N$ (the conditional distribution inside a cluster) there are weak correlations between x_i. Then $h(x_1, \ldots, x_N)$ would satisfy CLT when

$$\sum_1^N W_{0i} x_i \approx \mathcal{N}(\mu, \sigma^2)$$

for $\mu = \mu(\beta, \hat{W}), \sigma^2 = \sigma^2(\beta, \hat{W})$.

$$\mu = \langle h \rangle = \sum W_{0i} \langle x_i \rangle_N := \sum W_{0i} m_i$$

$$\begin{aligned} \sigma^2 = \langle h^2 \rangle_N - \langle h \rangle_N^2 &= \sum_{ij} W_{0i} W_{0j} (\langle x_i x_j \rangle_N - \langle x_i \rangle_N \langle x_j \rangle_N) \\ &\approx \sum W_{0i} (1 - \langle x_i \rangle_N^2) \approx \frac{1}{N} \sum (1 - \langle x_i \rangle_N^2) \\ &= 1 - \frac{1}{N} \sum \langle x_i^{(1)} x_i^{(2)} \rangle_N^2 = 1 - \left\langle \frac{x^{(1)} x^{(2)}}{N} \right\rangle \approx 1 - q \end{aligned}$$

For $x \sim \mu_{\beta,N}$ we have $h(x) \overset{d}{\approx} \mathcal{N}(\sum_1^N W_{0i} m_i, 1-q)$. Thus we obtain a nice joint distribution of $(x_0, h(x_1, \ldots, x_N))$ under $\mu_{\beta,N+1}$. We then calculate

$$\langle x_0 \rangle_{N+1} = \mathsf{Pr}[x_0 = 1] = \mathsf{Pr}[x_0 = -1] = \frac{e^\beta \mu - e^{-\beta\mu}}{e^\beta \mu + e^{-\beta\mu}} = \mathsf{tanh}(\beta\mu) = \mathsf{tanh}(\beta \sum W_{0i} \langle x_i \rangle_N)$$

$$\begin{aligned}\mathsf{Pr}[x_0 = x] &\propto \int e^{\beta h x} \mathsf{exp}(-\frac{(\beta-\mu)^2}{2\sigma^2}) dh \\ &= \int \mathsf{exp}(-\frac{(h-\mu-\beta\sigma^2 x)^2}{2\sigma^2}) \mathsf{exp}(\frac{-\mu^2 + (\beta\sigma^2 x + \mu)^2}{2\sigma^2}) dh \propto \mathsf{exp}(\beta\mu x)\end{aligned}$$

We can then solve for the overlay:

$$q = \mathbb{E}_W \frac{1}{N} \sum_{i=0}^{N+1} \langle x_i \rangle_{N+1}^2 = \mathbb{E}_W \langle x_0 \rangle_N^2 = \mathbb{E}\, \mathsf{tanh}(\beta \sum W_{0i} \langle x_0 \rangle_N) := \mathbb{E}\, \mathsf{tanh}(\beta \langle h \rangle_N)^2$$

where the $W_{0i}, \langle x_i \rangle_N$ are independent. We then get

$$\langle h \rangle \approx \mathcal{N}(0, \frac{1}{N} \sum \langle x_i \rangle_N^2) = \mathcal{N}(0, q)$$

So

$$q = \mathbb{E}\, \mathsf{tanh}(\beta t)^2 = \mathbb{E}_{t \sim \mathcal{N}(0,1)} \mathsf{tanh}(\beta \sqrt{q} \cdot t)^2$$

giving us an equation for the typical overlay from the replica method. To locate clusters

$$\langle x_0 \rangle_{N+1} = \mathsf{tanh}(\beta \sum W_{0i} \langle x_i \rangle_N) = \mathsf{tanh}(\beta \langle h \rangle_N)$$

almost works.

$$P_{N+1}(h) \propto \mathsf{cosh}(\beta h) P_N(h)$$

The normalizaing constant of $P_{W+1}(h)$ is

$$\int \frac{e^{\beta h} + e^{-\beta h}}{2} \cdot (\text{density of } \mathcal{N}(\langle h \rangle_N), 1-q) dh = e^{\frac{1-q}{2}\beta^2} \mathsf{cosh}(\beta \langle h \rangle_N)$$

Implying

$$p_{N+1}(h) = \frac{e^{\frac{-1-q}{2}\beta^2}}{\mathsf{cosh}(\beta \langle h \rangle_N)} \mathcal{N}(\langle h \rangle_N, 1-q)$$

Let $t := \mathsf{cosh}(\beta \langle h \rangle_N)$. Then

$$\begin{aligned}\langle h \rangle_{N+1} &= t \mathbb{E} h\, \mathsf{cosh}(\beta h) \\ &= t \mathbb{E}_{h \sim \mathcal{N}(0,1)} (h + \langle h \rangle_N) \mathsf{cosh}(\beta(\sqrt{1-q} \cdot h + \langle h \rangle_N)) \\ &= \langle h \rangle_N t \sqrt{1-q} \mathbb{E}_h \mathsf{cosh}(\beta(\sqrt{1-q} \cdot h + \langle h \rangle_N)) \\ &= \langle h \rangle_N t \beta (1-q) \mathbb{E}_h \mathsf{sin}(\beta h) \\ &= \langle h \rangle_N + \beta(1-q) \mathsf{tanh}(\beta \langle h \rangle_N) = \langle h \rangle_N + \beta(1-q) \langle x_0 \rangle_{N+1}\end{aligned}$$

Then

$$\langle h \rangle_N = \langle h \rangle_{N+1} - \beta(1-q)\langle x_0 \rangle_{N+1} = \sum J_{0i}\langle x_i \rangle_{N+1} - \beta(1-q)\langle x_0 \rangle_{N+1}$$

It follows

$$\langle x_0 \rangle = \mathsf{tanh}(\beta \sum w_{0i}\langle x_i \rangle - \beta^2(1-q)\langle x_0 \rangle)$$

Then for all i

$$\langle x_i \rangle = \mathsf{tanh}(\beta \sum w_{ij}\langle x_j \rangle - \beta^2(1-q)\langle x_i \rangle)$$

We can find the center of the these clusters with a nonlinear power method.