LECTURE 4

Continuous-time stochastic processes

Def: For $(\Omega, \mathcal{F}, \mathbb{P})$ prob. space, a process is a family of r.v.,
$(f(t) : \Omega \longrightarrow \mathbb{R})_{t \in T}$.
$T = \mathbb{Z}_{\geq 0}$: discrete time (rand. walk) $T = \mathbb{R}_{\geq 0}$ or $[0,1]$: cts. time

Often think of sample paths : given $f : T \times \Omega \longrightarrow \mathbb{R}$,
look at $f(\cdot, \omega)$, entire path for one outcome.

Generally, path $\in \{f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}\} =: P$.
Q: What subsets of P are measurable?
σ-algebra generated by $\{f : f(t) \in B\}$ for $B \subset \mathbb{R}$ Borel, $t \in \mathbb{R}$
$=$ product algebra $\mathcal{B}^T$.

Ex: $\{f : \sup_{t \in [0,1]} f(t) \leq 1\} \notin \mathcal{B}^T$. (All measurable sets have countable support.)
$\longrightarrow$ product algebra "weak". But, if replace P w/ $\{f \text{ continuous}\}$, this set is measurable!

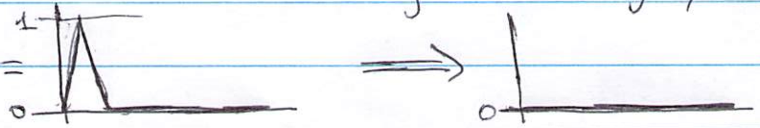
Like to work w/ path space P of f determined by, e.g., $(f(t))_{t \in \mathbb{Q}}$
$\longrightarrow$ makes product algebra good enough.

Def: f, g processes equal in finite distributions, $f \overset{(f.d.)}{=} g$, if
$\mathrm{Law}((f(t_1), \ldots, f(t_k))) = \mathrm{Law}((g(t_1), \ldots, g(t_k)))$.

Fact: (Corollary of Kolmogorov ext. thm.) If $f \overset{(f.d.)}{=} g$, then $\forall A \in \overbrace{\mathcal{B}^T}^{\text{product alg.}}$,
$\mathbb{P}[f \in A] = \mathbb{P}[g \in A]$ ("laws equal w.r.t. product alg.")

Def: $f_n \xrightarrow{(f.d.)} f$ if $(f_n(t_1),\dots,f_n(t_k)) \Rightarrow (f(t_1),\dots,f(t_k))$

Rk: "Weak" notion of convergence. E.g., deterministic functions

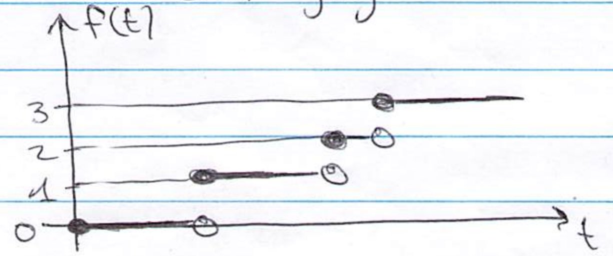
$f_n =$  $= f$

even though, e.g., $\underbrace{\sup_{t\in[0,1]} f_n(t)}_{=1} \not\Rightarrow \underbrace{\sup_{t\in[0,1]} f(t)}_{=0}$

But: ① sometimes already useful; ② sometimes implies stronger forms of convergence w/ extra conditions.

Def: Counting process is $f(t)$ with paths satisfying:

- $f(t) \in \mathbb{Z}_{\geq 0}$
- $f(t)$ non-decreasing
- $f(t)$ right-continuous (cadlag)



Ex: If $A \subset \mathbb{R}_{\geq 0}$ is random countable set of separated pts, then $f(t) := |A \cap [0,t]|$ is counting process. → point process.

Def: Bernoulli (counting) process $f \sim BerP(n,\lambda)$: $X_i \overset{iid}{\sim} Ber\left(\frac{\lambda}{n}\right)$, $A := \{\frac{i}{n} : X_i = 1\}$, $f(t)$ built from A.

Def: (PPP, model 1) For each $k \geq 0$, draw $N_k \sim Pois(\lambda)$, let $A_k := \{N_k \text{ pts. uniformly random in } [k,k+1]\}$, $A := \bigcup_{k\geq 0} A_k$, $f(t) \sim PPP(\lambda)$ built from this A.

Thm: $f_n \sim BerP(n,\lambda) \xrightarrow{(f.d.)} f \sim PPP(\lambda)$.