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LECTURE 9

Recall martingale betting strategy: can view differently as

$$\tilde{M}_n := \sum_{i=1}^n X_i \cdot 2^{i-1} \quad \text{for } X_i \sim \mathrm{Unif}(\{\pm 1\})$$

"stopped" at first time that $X_n = +1$.

$$T := \min\{n : X_n = +1\}, \quad M_n := \tilde{M}_{T \wedge n}$$

$a \wedge b := \min\{a, b\}$.

$\tilde{M}_n$ is a martingale (weighted RW); saw that M_n is, too.

__Def:__ $(\mathcal{F}_n)$ filtration $\rightsquigarrow T : \Omega \to \{0, 1, 2, \dots\} \cup \{+\infty\}$ is a __stopping time__ if $\{\omega : T(\omega) = n\} \in \mathcal{F}_n \ \forall n$

equivalently: $\{\omega : T(\omega) \le n\} \in \mathcal{F}_n \ \forall n$

__Ex:__ $S_n = \sum_{i=1}^n X_i \longrightarrow T := \min\{n : S_n = a\}$ "hitting time"

$T := \min\{n : S_n \in \{a, b\}\}$ "exit time"

__Thm:__ If T is stopping time, (M_n) (sub/super) martingale, then $(M_{T \wedge n})$ (sub/super) mgl.

__Pf:__ $H_n := \mathbb{1}\{T \ge n\}$ is predictable, since $1 - H_n = \mathbb{1}\{T \le n-1\} \in \mathcal{F}_{n-1}$

$$(H \cdot M)_n = \sum_{i=1}^n H_i (M_i - M_{i-1}) = \sum_{i=1}^{T \wedge n} (M_i - M_{i-1})$$

$$= M_{T \wedge n} - M_0.$$

$H \ge 0$, predictable $\rightsquigarrow$ preserves mgl, sub/super. Adding M_0 has no effect on these properties.

__Cor:__ If T is __bounded__ stopping time ($T \le t$ a.s.) then $\mathbb{E} M_T = \mathbb{E} M_0$ for (M_n) mgl.

__Q:__ Does this happen more generally?

Ex: $(M_n) = \text{SRW}$, $T := \min\{n : M_n = 1\}$. Will see $T < \infty$ a.s., but $\mathbb{E} M_T = 1 \neq 0 = \mathbb{E} M_0$, while $\mathbb{E} M_{T \wedge n} = 0 \ \forall n$.

<u>Preliminary Q</u>: M_T is m'able w/r/t which σ-algebra?

Def: For $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots \subseteq \mathcal{F}$ filtration, T stopping time,
$\mathcal{F}_T := \{A \in \mathcal{F} : \forall n \geq 0,\ A \cap \{T \leq n\} \in \mathcal{F}_n\}$.
Intuition: "info up to <u>random</u> time T"
A s.t. "if T happened I can decide if A happened"

Prop: T is $\mathcal{F}_T$-m'able, for (M_n) adapted, M_T is $\mathcal{F}_T$-m'able.

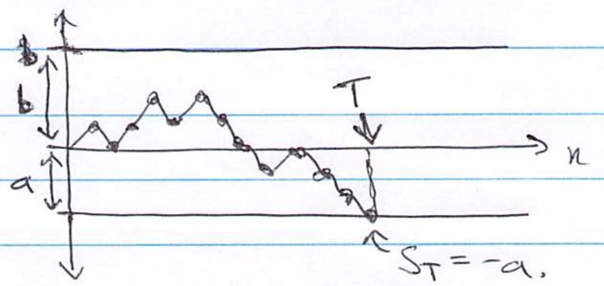
Thm: (Doob Optional Stopping) (M_n) mgl, T stopping time. Suppose
(1) $T \leq t$ a.s. for some $t \geq 0$ OR ("bdd in time")
(2) $T < \infty$ a.s. and $|M_n| \leq C$ a.s. OR ("bdd in space")
(3) $\mathbb{E} T < \infty$ and $|M_n - M_{n-1}| \leq C$ a.s.
Then, $M_T \in L^1$ and $\mathbb{E} M_T = \mathbb{E} M_0$.

Pf: (1): $(M_{T \wedge n})$ mgl, take $n = t \rightsquigarrow \mathbb{E} M_T = \mathbb{E} M_0$.
(2): $M_{T \wedge n} \longrightarrow M_T$ a.s., uniformly bounded by C.
dominated convergence $\Rightarrow$ $\underset{\mathbb{E} M_0}{\underset{\parallel}{\mathbb{E} M_{T \wedge n}}} \longrightarrow \mathbb{E} M_T$

(3): $|M_{T \wedge n} - M_0| = \left|\sum_{i=1}^{T \wedge n} (M_i - M_{i-1})\right| \leq C (T \wedge n) \leq CT$.
T integrable $\rightarrow$ again by dom. conv.
$\underset{0}{\underset{\parallel}{\mathbb{E}(M_{T \wedge n} - M_0)}} \longrightarrow \mathbb{E}(M_T - M_0) = \mathbb{E} M_T - \mathbb{E} M_0$.

③

Ex: $S_n = \text{SRW}$, martingale
$T := \min\{n : S_n \in \{-a, b\}\}$



Q: How long to exit? Which side?

Prop: $\mathbb{E}T < \infty$.

Pf: If $X_{k+1} = \cdots = X_{k+(a+b)} = +1$, then T has definitely happened. $\mathbb{P}[\text{run of } (a+b) +1\text{'s}] = 2^{-(a+b)}$.
$Y_1, Y_2, \ldots \sim \text{Ber}(2^{-(a+b)})$, $Y_i = \mathbb{1}\{i^{th} \text{ run all} = +1\}$
$G := \min\{k : Y_k = 1\} \rightsquigarrow T \le (a+b)G$
$G \sim \text{Geom}(2^{-(a+b)})$ ↪ cf. "stochastic domination".

Def: $G \sim \text{Geom}(p)$ has $\mathbb{P}[G = k] = (1-p)^{k-1} p$ for $k = 1, 2, \ldots$.
$\rightsquigarrow \mathbb{E}G = 1/p$.

$\Longrightarrow \mathbb{E}T \le (a+b)2^{a+b} < \infty$.

$\rightsquigarrow$ OST (3) applies to $S_T \rightsquigarrow \mathbb{E}S_T = \mathbb{E}S_0 = 0$.

$p := \mathbb{P}[\text{"exit upwards"}] = \mathbb{P}[S_T = b]$, $1-p = \mathbb{P}[S_T = -a]$.

$0 = \mathbb{E}S_T = p \cdot b + (1-p)\cdot(-a) \Longrightarrow p(a+b) = a$
$\Longrightarrow p = \frac{a}{a+b}$.

Cor: $\mathbb{P}[\text{exit at } b] = \frac{a}{a+b}$, $\mathbb{P}[\text{exit at } -a] = \frac{b}{a+b}$.

__Idea:__ Build further "derived martingales" to gather more info about exit time.

Prop: S_n SRW $\rightsquigarrow$ $M_n := S_n^2 - n$ is martingale.

Pf: $\mathbb{E}[S_n^2 - n \mid \mathcal{F}_{n-1}] = \mathbb{E}[(S_{n-1} + X_n)^2 - n \mid \mathcal{F}_{n-1}]$

$$= \mathbb{E}[S_{n-1}^2 + 2X_nS_{n-1} + X_n^2 - n \mid \mathcal{F}_{n-1}]$$

$$= S_{n-1}^2 - (n-1) + 2S_{n-1}\cancelto{0}{\mathbb{E}[X_n \mid \mathcal{F}_{n-1}]} + \cancelto{1}{\mathbb{E}[X_n^2 \mid \mathcal{F}_{n-1}]} - \cancel{1}.$$

$\implies (M_{T\wedge n})$ martingale. Analyze increments:

$|M_{T\wedge n} - M_{T\wedge(n-1)}| = 0$ if $T \le n-1$, otherwise

$$= |M_n - M_{n-1}| = |2S_nX_n + 1|$$

$$\le 2|S_n| + 1 \le 2\, a\vee b + 1 < \infty \quad (\text{since } T \ge n).$$

Rk: Increments of M_n itself not bounded! Need this restriction.

OST (3) $\implies \mathbb{E}M_{T\wedge T} = \mathbb{E}M_T = \mathbb{E}M_0 = 0.$

$$0 = \mathbb{E}[S_T^2 - T] = \mathbb{E}S_T^2 - \mathbb{E}T$$

$$= a^2\cdot\frac{b}{a+b} + b^2\cdot\frac{a}{a+b} - \mathbb{E}T = ab - \mathbb{E}T.$$

Rk: Can determine full joint law like this!

Cor: $\mathbb{E}[\text{time of exit of } [-a,b]] = ab.$

Sequence of polynomial martingales gives more info on $\mathrm{Law}((\overset{\text{time to exit}}{T}, \overset{\text{side of exit}}{S_T}))$.

Prop: $M_n := S_n^3 - 3nS_n$ is a martingale and OST can be applied w/ stopping time T.

$$\mathbb{E}T = \mathbb{E}T\mathbb{1}\{S_T = -a\} + \mathbb{E}T\mathbb{1}\{S_T = b\}$$

$\implies$ Can compute $\mathbb{E}TS_T = -a\,\mathbb{E}T\mathbb{1}\{S_T=-a\} + b\,\mathbb{E}T\mathbb{1}\{S_T = b\}$

$\implies$ Solve for $\mathbb{E}T\mathbb{1}\{S_T = -a \text{ or } b\} \rightsquigarrow \mathbb{E}[T \mid \text{exit upwards or downwards}].$